A multifractal model for the velocity gradients dynamics in turbulent flows

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Velocity Gradients

Lagrangian Dynamics

$$\boxed{A_{ij} \equiv \frac{\partial u_i}{\partial x_j}}$$

- Navier-Stokes equation

$$\frac{\mathrm{d}A_{ij}}{\mathrm{d}t} = -A_{ik}A_{kj} - \frac{\partial^2 p}{\partial x_i \partial x_j} + \nu \nabla^2 A_{ij}$$

- Qualitative and quantitative local behaviour
- Applications (e.g., particle transport)
- Statistical properties from experiments and/or DNS
  vorticity alignment, RQ-plane, intermittency...

Modelling

$$\frac{\mathrm{d}A_{ij}}{\mathrm{d}t} = -A_{ik}A_{kj} - \underbrace{\frac{\partial^2 p}{\partial x_i \partial x_j} + \nu \nabla^2 A_{ij}}_{\text{need modelling}}$$

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- Restricted Euler (Vieillefosse 1982) → isotropic pressure, no viscosity

$$\frac{\mathrm{d}A_{ij}}{\mathrm{d}t} = -A_{ik}A_{kj} + \frac{1}{3}\mathrm{Tr}(\mathsf{A}^2)\delta_{ij}$$

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- analytical solution (Cantwell 1992)
- vorticity alignments
- finite time singularity

Modelling

$$\frac{\mathrm{d}A_{ij}}{\mathrm{d}t} = -A_{ik}A_{kj} + \frac{1}{3}Tr(A^2)\delta_{ij} - \frac{A_{ij}}{\tau_0}$$

- Linear damping model (Martin et al. 1998)

Modelling

$$\frac{\mathrm{d}A_{ij}}{\mathrm{d}t} = -A_{ik}A_{kj} + \frac{1}{3}Tr(A^2)\delta_{ij} - \frac{A_{ij}}{\tau(\mathsf{A})}$$

- Linear damping model (Martin et al. 1998)
- Lagrangian linear damping (Jeong & Girimaji 2003)
  $\rightarrow$ regularizes singularity

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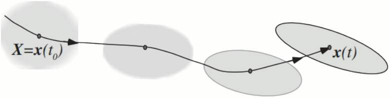
- Linear damping model (Martin et al. 1998)
- Lagrangian linear damping (Jeong & Girimaji 2003)
  → regularizes singularity
- More sophisticated models for the pressure
  Tetrad model (Chertkov et al. 1999)
  → stochastic models

Modelling

- Recent fluid deformation (RFD) approximation (Chevillard & Meneveau 2006)



$$\mathrm{d}A_{ij} = V_{ij}^{\mathrm{RFD}}\mathrm{d}t + D_{ijkl}\mathrm{d}W_{kl}$$

- Time scales τ_η and $T \to R_e \sim (T/\tau_\eta)^2$
- Quite realistic for moderate R_e

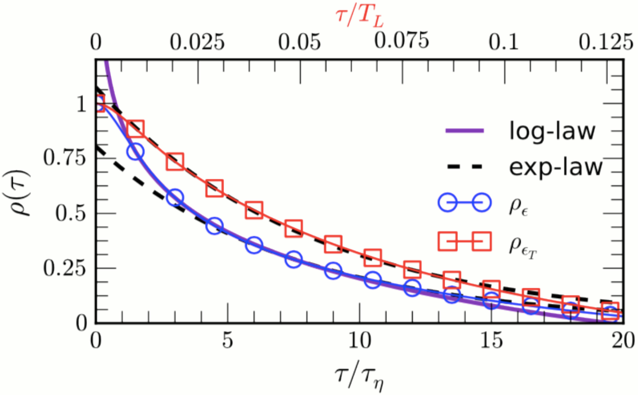
Problems

- Break down for $R_e \gg 1$
- Intermittency
  - A_{ij} correlated over τ_η
  - $|A_{ij}|$ over T
- Multifractality missing
  - $\langle \epsilon^q \rangle \rightarrow$ power-laws of R_e
  - $\langle \ln \epsilon(t) \ln \epsilon(t+\tau) \rangle \sim \ln T/\tau$

Remark

$$\langle \ln \epsilon(t) \ln \epsilon(t+\tau) \rangle \sim \ln(T/\tau) \;? \qquad \text{(Arneodo et al. 1998)}$$

$$\sim \exp(-\tau/T) \;? \qquad \text{(Yeung \& Pope 1989)}$$



(Huang & Schmitt, JFM 2014)

Girimaji-Pope model

- dissipation $\epsilon \approx$ log-normal

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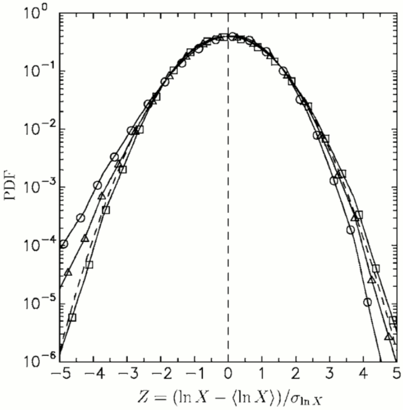
- dissipation $\epsilon \approx$ log-normal
- pseudo-dissipation $\varphi \equiv \nu A_{ij} A_{ij}$ *remarkably* log-normal (Pope 1988)



(Yeung et al., Phys. Fluids 2006)

Girimaji-Pope model

- Ornstein-Uhlenbeck process for $\ln(\varphi)$

$$\mathrm{d}\varphi(t) = \varphi(t)\left[\hat{a}^2 - \frac{1}{T}\ln\left(\frac{\varphi(t)}{\mathbb{E}[\varphi]}\right)\right]\mathrm{d}t + 2\,\hat{a}\,\varphi(t)\,\mathrm{d}W(t)$$

- K41: $\mathbb{E}[\varphi] = 1/{\tau_\eta}^2$

- $Re = (T/\tau_\eta)^2$ and $\hat{a}$ a free parameter

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- $\mathbb{E}[\ln\varphi(t)\ln\varphi(t+\tau)] \sim e^{-\tau/T}$

Girimaji-Pope model

- $\mathbb{E}\left[\varphi^q\right] \sim \exp\left\{\hat{a}^2 T q(q-1)\right\}$

  power-laws of $R_e \quad \Rightarrow \quad \hat{a}^2 T \sim \ln(R_e)$

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- Problem: coarse-grained pseudo-dissipation

$$\varphi_\tau \equiv \frac{1}{\tau}\int_{t-\tau}^{t} \varphi(s)ds$$

$$\frac{\mathbb{E}[\varphi_\tau^q]}{(\mathbb{E}[\varphi])^q} \sim \exp(\alpha_q\, \hat{a}^2 T) \;\Rightarrow\; \text{diverge as } Re \to \infty$$

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- Inconsistent with the Refined Similarity Hypothesis

Girimaji-Pope model

- From φ to A_{ij} (Girimaji & Pope 1990)
- strategy: build general equation for A_{ij}

$$\mathrm{d}A_{ij}(t) = \left[(RE)_{ij} + M_{ij}\right]\mathrm{d}t + D_{ijkl}\,\mathrm{d}W_{kl}(t)$$

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- multiplicative noise

Multifractal model

Gaussian multiplicative chaos

- Success in Eulerian turbulence (RMP, Garban & Chevillard 2016)

- Lagrangian version

$$\mathrm{d}X_{\tau_\eta}(t) = \left[\beta_{\tau_\eta}(t) - \frac{1}{T}X_{\tau_\eta}(t)\right]\mathrm{d}t + \frac{1}{\sqrt{\tau_\eta}}\mathrm{d}W(t)$$

- Regularized fractional Gaussian noise with $H = 0$

$$\beta_{\tau_\eta}(t) = -\frac{1}{2}\int_{-\infty}^{t}\frac{1}{(t-s+\tau_\eta)^{3/2}}\mathrm{d}W(s)$$

Gaussian multiplicative chaos

- Solution: standard OU + fractional OU with $H = 0$
- $\mathbb{E}\left[\left(X_{\tau_\eta}\right)^2\right] \underset{\tau_\eta \to 0}{\sim} \ln\left(\frac{T}{\tau_\eta}\right)$

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$$\mathbb{E}\left[X(t)X(t+\tau)\right] \underset{\tau_\eta \to 0}{\sim} \ln\left(\frac{T}{\tau}\right)$$

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- Exactly multifractal model for pseudo-dissipation

$$\boxed{\varphi_{\tau_\eta} \equiv \frac{1}{\tau_\eta^2}\, e^{\gamma X_{\tau_\eta} - \gamma^2 \mathbb{E}\left[\left(X_{\tau_\eta}\right)^2\right]}}$$

Asymptotic results

- Power-law moments

$$\mathbb{E}\left[\varphi_{\tau_\eta}^q\right] \sim \left(\frac{T}{\tau_\eta}\right)^{\frac{\gamma^2}{2}q(q-1)}$$

- Finite moments of coarse-grained pseudo-dissipation

$$\frac{\mathbb{E}\left[\varphi_\tau^q\right]}{(\mathbb{E}\varphi_{\tau_\eta})^q} \sim \left(\frac{T}{\tau}\right)^{\frac{\gamma^2}{2}q(q-1)}$$

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- $\mathrm{d}\varphi_{\tau_\eta}$ from $\mathrm{d}X$ with Itō

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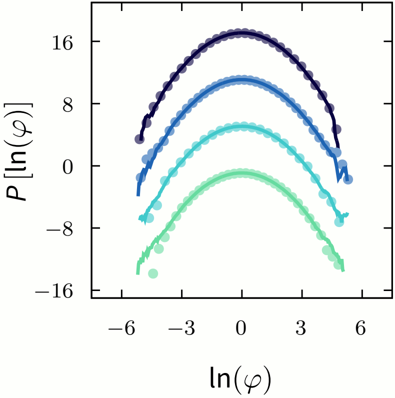
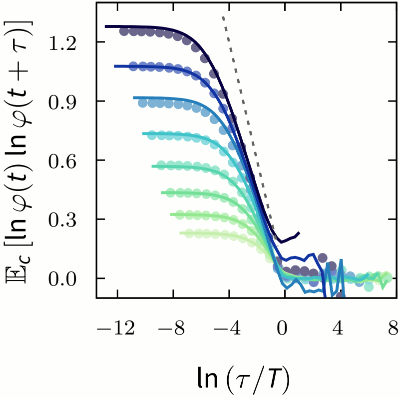
- It fails.
- $\beta_{\tau_\eta}(t) \rightarrow \hat{\beta}_{\tau_\eta}(t) = -\frac{1}{2}\int_{-\infty}^{t} \frac{1}{(t-s+\tau_\eta)^{3/2}} \frac{A_{ij}(s)}{\sqrt{\varphi(s)}} \mathrm{d}W_{ij}(s)$
- No analytical results even for φ $\rightarrow$ simulations

Numerics

Comparing φ and φ_{τ_η}

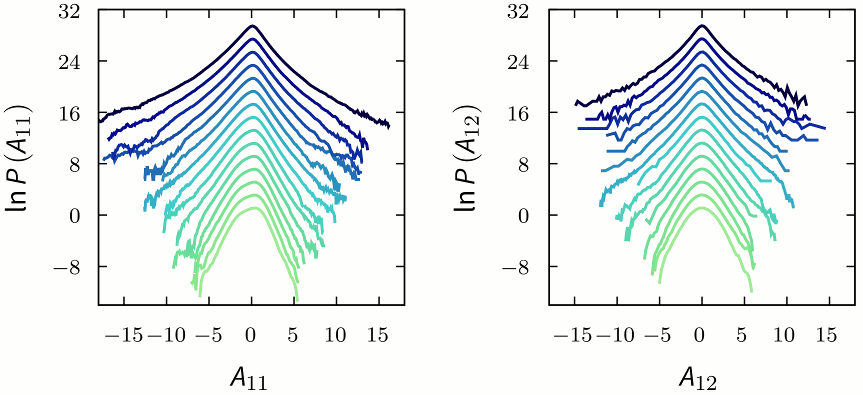
- PDF and correlation





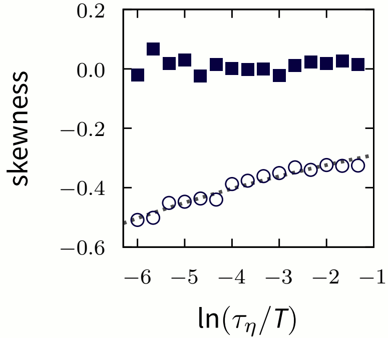
Resulting A_{ij}

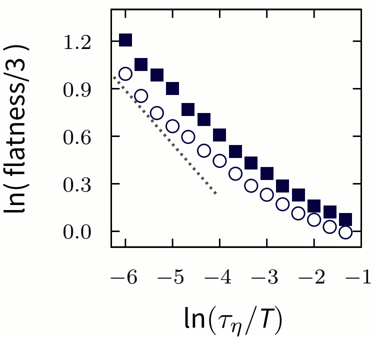
- PDFs



Resulting A_{ij}

- Scaling

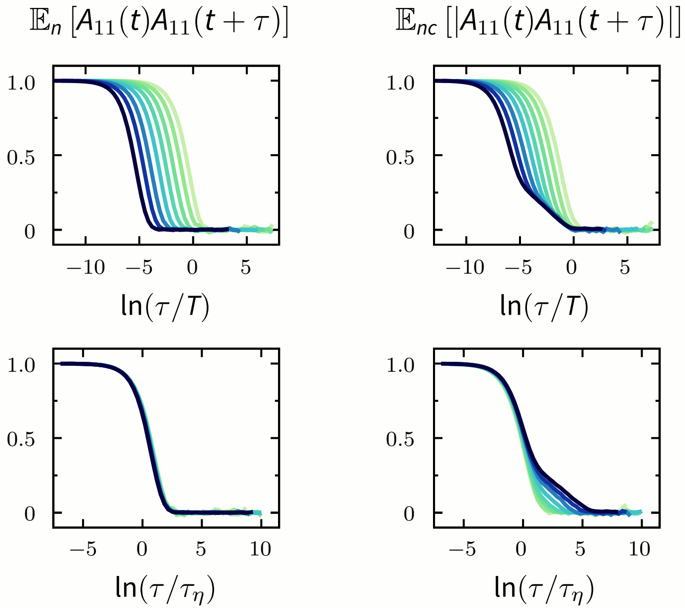




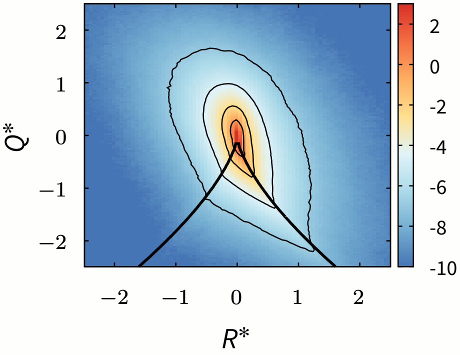
$$\frac{\mathbb{E}[A_{11}^3]}{(\mathbb{E}[A_{11}^2])^{3/2}} \propto -\left(\frac{T}{\tau_\eta}\right)^{0.11}$$

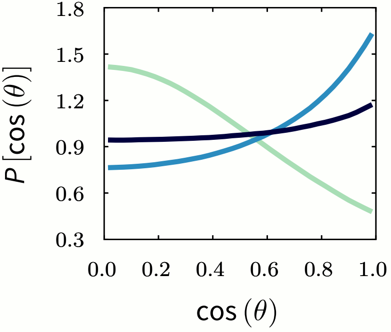
$$\frac{\mathbb{E}[A_{11}^4]}{(\mathbb{E}[A_{11}^2])^{2}} \propto \left(\frac{T}{\tau_\eta}\right)^{0.34}$$

Resulting A_{ij}



Further properties





Conclusion

- Realistic and multifractal model for Lagrangian isotropic turbulence
- Based on a new rigorous multifractal process
- Solves issues of the RFD model
  - Works for arbitrarily large Reynolds number
  - Correct correlation time scales of A_{ij}

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- What now?
  - further improvements
  - analytical understanding of $\beta(t) \rightarrow \hat{\beta}(t)$