Vorticity measures and the limit of vanishing viscosity in the presence of boundaries

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How to describe? Weak solutions as a tool to describe limiting flow

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$\nu > 0$: ν-Navier-Stokes;

$\nu = 0$: Euler.

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OBS. Actually, iff.

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Use ∂ condition for ω^ν to **derive L^1-bounds for ω^ν**; improve convergence to vortex sheets in limit $\nu \to 0$ from w-$*$ $(H^1)'$ to w-$*$ $\mathcal{BM}$.

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- For each $\Phi \in C_c^\infty$, $\operatorname{div} \Phi = 0$,

$$\int_0^T \int_\Omega \partial_t \Phi \cdot u + D\Phi : u \otimes u = 0.$$

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Say $\omega \in L^\infty((0,T);\mathcal{BM}(\mathbb{R}^2) \cap H^{-1}_{\rm loc}) \cap C^0([0,T);H^{-M}(\mathbb{R}^2))$, $M > 0$, is a *weak solution* of the weak vorticity formulation of Euler if, for all $\varphi \in C^\infty_c([0,T) \times \mathbb{R}^2)$:

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Obs. Similar proof if $\omega_0 \in \mathcal{BM}_{+,c} \cap H^{-1}_{\text{loc}} + L^1_c(\mathbb{R}^2)$ (Delort, 1991; Vecchi, Wu, 1993)

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Then u^∞ *weak Euler solution.*

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Step 2

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$$\int_0^T \int_\Omega \int_\Omega \nabla\varphi(t,x)\cdot K_\Omega(x,y)\,(1-\chi(y))\omega^\infty(t,y)\chi(x)\omega^\infty(t,x)\,dxdydt$$

$$= -\int_0^T \int_\Omega D\Phi(t,x)u^\infty(t,x)\otimes u^\infty(t,x)\,dxdt,$$

any smooth cutoff χ, $\chi \equiv 1$ on suppφ.

Step 2 u^∞ weak solution of velocity equation

Need *simply connected* for Step 2:

$\Phi \in C_c^\infty$, div $\Phi = 0 \iff \Phi = \nabla^\perp \varphi$, $\varphi \in C_c^\infty$

Mollify ω^∞ and move away from ∂, then pass to limit in mollification and distance to ∂ parameters

Show that

$$\int_0^T \int_\Omega \int_\Omega H_\Omega^\varphi(t,x,y)\, \chi(y)\omega^\infty(t,y)\chi(x)\omega^\infty(t,x)\, dxdydt +$$

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Solve ν-NS with initial data u_0^ν, no slip BC.

Then $u^\nu \to u^\infty$ strong $L^\infty(0, T; L^2)$, Then $u^\nu \to u^\infty$ strong $L^\infty(0,T;L^2)$,

$\|\omega^\nu\|_{L^1(\Omega)} \leq C$, all ν, Then $u^\nu \to u^\infty$ strong $L^\infty(0, T; L^2)$,

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OBS. $\omega^\nu \rightharpoonup \omega^\infty - \pi\delta_{\{|x|=1\}}$ weak-$*$ $L^\infty(0,T;\mathcal{BM}(\overline{\Omega}))$.

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(6) Connect to recent criterion by Theo Drivas and Huy Nguyen; hypothesis on structure function for velocity. Equivalent to ω^ν bounded in $L^2(0, T; H^{-1+\delta})$.

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