inviscid limit, the particles must stick upon collisions. for sufficiently smooth initial data, it may be possible to construct the solution from the Lagrangian manifold (2.8, 2.9) or its multidimensional generalization, by just searching the maximum, for a given $\mathbf{x}$, of the finitely many branches present.

3 The Fourier–Lagrange representation and artefacts

In this section we show that formal manipulations of the inviscid Burgers equation with random initial conditions, even though they include apparently terms of all orders, can nevertheless lead to completely incorrect results, *e.g.* for the energy spectrum. This section is entirely based on work by Fournier and Frisch [23]. The theory is given in one dimension but similar results can be established in higher dimensions.

In one dimension, it follows from (2.10, 2.11), that the Eulerian solution to the initial value problem for the decaying Burgers has the following implicit representation:

$$\begin{aligned} u(x,t) &= u_0(a) \\ x &= a + t u_0(a). \end{aligned} \quad (3.1)$$

This becomes explicit if, instead of working with $u(x,t)$, we use its spatial Fourier transform (2π-periodicity is assumed for convenience)

$$\hat{u}(k,t) \equiv \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-ikx} u(x,t)\,\mathrm{d}x \quad (3.2)$$

and make the change of variables $x \mapsto a$, to obtain

$$\hat{u}(k,t) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-ikx(a,t)}\, u_0(a)\frac{\partial x}{\partial a}\,\mathrm{d}a, \quad x(a,t) \equiv a + t u_0(a). \quad (3.3)$$

Equation (3.3) is called the *Fourier–Lagrangian* representation. A first integration by parts yields

$$\hat{u}(k,t) = \frac{1}{2\pi}\frac{1}{ik}\int_0^{2\pi} \mathrm{e}^{-ik(a+tu_0(a))}\, u_0'(a)\,\mathrm{d}a. \quad (3.4)$$

A second integration by parts leads then to

$$\hat{u}(k,t) = \frac{1}{2\pi}\frac{1}{ikt}\int_0^{2\pi} \mathrm{e}^{-ik(a+tu_0(a))}\,\mathrm{d}a, \quad k \neq 0. \quad (3.5)$$

If we now take random homogeneous Gaussian initial conditions, we can easily calculate moments of $\hat{u}(k,t)$ because they just involve averages of exponentials having the Gaussian initial velocity in their arguments. For example, the energy spectrum, related to the correlation function by

$$\langle \hat{u}(k,t)\hat{u}(k',t)\rangle = E(k,t)\delta_{k,k'}, \tag{3.6}$$

where $\delta_{k,k'}$ is a Kronecker delta, has the following expression

$$E(k,t) = \frac{1}{2\pi}\frac{1}{k^2t^2}\int_0^{2\pi} \mathrm{e}^{-ikh}\mathrm{e}^{-\frac{1}{2}k^2t^2S_2(h,0)}\,\mathrm{d}h, \tag{3.7}$$

where $S_2(h,0) \equiv \left\langle [u_0(h) - u_0(0)]^2 \right\rangle$ is the second-order structure function of the initial velocity field. If the latter is smooth, as we shall assume, we have $S_2(h,0) \propto h^2$ for $h \to 0$. It then follows by a simple Laplace-type asymptotic expansion of (3.7) that

$$E(k,t) \propto k^{-3} \quad \text{when} \quad k \to \infty. \tag{3.8}$$

This is obviously the wrong answer: for Gaussian initial conditions there will be shocks with a non-vanishing probability for any $t > 0$. Their signature is a k^{-2} law in the energy spectrum at high wavenumbers, as shown in Section 2.2.

What went wrong? After the appearance of the first shock the Lagrangian map $a \mapsto x$ is not monotonic and the change of variable from (3.2) to (3.3) is valid only outside of the Lagrangian shock interval. Hence, in (3.3) we should excise this interval from the domain of integration. If we do not remove it, we are actually calculating the Fourier transform of a function obtained by superposing the threee branches shown in Figure 10 with a plus sign for the two direct branches and a minus sign for the retrograde branch (the sign comes from the lack of an absolute value on the Jacobian $\partial x/\partial a$ in (3.3)). Obviously, this superposition has two square-root cusps as shown in Figure 10. This produces $k^{-3/2}$ tails in the Fourier transform and, hence, explains the spurious k^{-3} energy spectrum. Note also that this superposition of three branches is not a solution to the Burgers equation, the latter being nonlinear. This phenomenon is not related to the well known non-uniqueness of the solution to the Burgers equation with zero viscosity without proper additional conditions [24].

The problem is actually worse than suggested so far. It is easily shown that if the the initial velocity is deterministic and smooth, the function of the time defined by (3.3), for *fixed* wavenumber k, is entire, that is, its Taylor series around $t = 0$ has an infinite radius of convergence. There is no way to see the time $t_\star$ of the first preshock from this function. A preshock is indeed an "ultraviolet" singularity which is not seen in the temporal behavior of a single spatial Fourier component. This result has an important

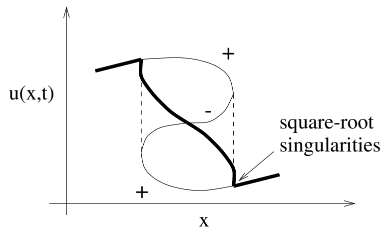

Fig. 10. Spurious solution of Burgers equation when three branches of a multi-valued solution are combined into one.

consequence for the case of random Gaussian initial conditions. Suppose we simply ignore the viscosity in the Burgers equation and expand the solution to all orders in a temporal Taylor series around $t = 0$ and then calculate various correlation functions and use Feynman graphs for bookkeeping of all the terms generated from averaging. We then find that the whole set can be resummed exactly and gives a spectrum with a k^{-3} tail. Of course, the origin of the "resummation miracle" is the Fourier–Lagrangian representation.

4 The law of energy decay

An important issue in burgulence and turbulence is that of the law of decay at long times when the viscosity is very small. Before turning to the Burgers equation let us recall a few things about the Navier–Stokes case. It is generally believed that high-Reynolds number turbulence has universal and non-trivial small-scale properties. In contrast, large scales, important for practical applications such as transport of heat or pollutants, are believed to be non-universal. This is however so only for the toy model of turbulence maintained by prescribed large-scale random forces. Very high-Reynolds number turbulence, decaying away from its production source, and far from boundaries can relax under its internal nonlinear dynamics to a (self-similarly evolving) state with universal and non-trivial statistical properties *at all scales*. Kármán and Howarth [31], investigating the decay of high-Reynolds number, homogeneous isotropic three-dimensional turbulence, proposed a self-preservation (self-similarity) ansatz for the spatial correlation function of the velocity: the correlation function keeps a fixed functional shape; the integral scale $L(t)$, characteristic of the energy-carrying eddies, grows in time and the mean kinetic energy $E(t) = u^2(t)$ decays, both following power laws; there are two exponents which can be related by the condition that the energy dissipation per unit mass $|\dot{E}(t)|$ should be proportional to u^3/L. But *an additional relation* is needed to actually determine the exponents. The invariance in time of the energy