Taylor Diffusion

Integrating in time a weakly stationary stochastic process yields a diffusive behavior at long time. This phenomenon is known as “Taylor Diffusion”.

1. An explicit example: Time-Integrated O.U:

Let $(X(t), t \geq 0)$ be the time integral of a stationary Ornstein-Uhlenbeck process. Explicitly,

$$X(t, \varpi) := \int_0^t v(t, \varpi) dt$$

with $(v(t), t \geq 0)$ a stationary Ornstein-Uhlenbeck process, namely a (1D) Gaussian centered stochastic process with correlation: $C(t) := C_0 e^{-|t|/\tau}$.

(i) Compute $\frac{d}{dt}\left\langle X^2 \right\rangle$.

(ii) Study the asymptotics $t \ll \alpha^{-1}$ and $t \gg \alpha^{-1}$.

2. General case

Let $(v(t), t \geq 0)$ be a centered weakly stationary stochastic process with correlation function C, such that

$$\int_0^\infty d\tau |C(\tau)| < \infty.$$

Let $X(t, \varpi) := \int_0^t dt\, v(t, \varpi)$.

Show that $\frac{d}{dt}\left\langle X^2 \right\rangle$ has a finite limit as $t \to \infty$, and relate it to the diffusion coefficient $D := \int_0^\infty d\tau C(\tau)$. This is Taylor's formula, which describes an *asymptotic* emergence of a diffusive behavior.

3. Diffusive behavior from a deterministic dynamics

Consider the deterministic standard map, describing the evolution of the pairs of variables (J, θ) as:

$$J(t+1) = J(t) + K \sin\theta \text{ and } \theta(t+1) = \theta(t) + J(t)$$

with $J(t=0) \sim U(0, 2\pi)$ and $\theta(t=0) \sim U(0, 2\pi)$ independent (uniform) random variables.

(i) Under the “random phase approximation”, that consists in assuming that for t large enough, the process $(\sin(\theta(t)), t \geq 0)$ has a vanishing correlation time, show that $< J >= 0$ and $< J^2 >\sim 2DT$ with $D = K^2/4$.

(ii) Check the quality of the approximation with numerical simulations, depending on the value of K.