Exercises involving Gaussian processes

Let us recall that a (1D) real-valued stochastic process $X(t,\varpi)$ is **Gaussian** *iff*
For all $k \in \mathbb{N}$ and all k-uples $(t_1 \cdots t_k) \in \mathbb{R}^k$ there exists a vector $\boldsymbol{\mu} \in \mathbb{R}^k$ and positive-definite symmetric matrix $\boldsymbol{C} \in \mathbb{R}^{k\times k}$ such that

$$\mathbf{X} := (X_{t_1} \cdots X_{t_k}) \underset{\text{law}}{\sim} \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{C}), \tag{1}$$

That is :

$$P\left[\bigcap_{i=1}^{k} X(t_i) \in dx_i\right] = \left(\prod_{i=1}^{k} dx_i\right) \frac{e^{-\frac{1}{2}\tilde{\mathbf{X}}^T \boldsymbol{C}^{-1}\tilde{\mathbf{X}}}}{(2\pi)^{k/2}(\det \boldsymbol{C})^{1/2}} \text{ with } \tilde{\mathbf{X}} := \mathbf{X} - \boldsymbol{\mu} \tag{2}$$

Ex. 1:
Show that Definitions 1 and 2 given in the course provide an equivalent characterization of Brownian motion.

Ex. 2:
Find the joint PDF for the Brownian motion with drift $\mathbf{X}_t := \mu + \sqrt{2\kappa}W_t$, where W_t denote the 1D canonical BM.

Ex. 3:
Check that BM indeed posesses inversion, rescaling, time reversal and time inversion symmetries.

Ex. 4:
Compute the moments $\langle W_t^p \rangle$ of 1D BM.

Hint: Show that characteristic function $\xi_X(t) := \left\langle e^{itX} \right\rangle$ of a 1D Gaussian random variable $X \sim \mathcal{N}(0,1)$ is $\chi_X(t) = e^{-t^2/2}$.

Ex. 5:
Let X_t, Y_t denote two independent (1D) Gaussian stochastic processes. Show that the following stochastic processes are themselves Gaussian:

(i) $\tilde{X}_t := X_{f(t)}$, where f denotes a strictly increasing function.

(ii) $\tilde{X}_t := X_t + \alpha Y_t$, where $\alpha \in \mathbb{R}$

Ex. 6:
Let W denotes the 1D canonical BM.

Study the following processes. (Stationarity ? Gaussianity ? Average ? Covariance ? Spectrum ?)
(i) $X_t := e^{-t}W(e^{2t}), t \geq 0$ (Ornstein-Uhlenbeck)
(ii) $X_t := W_t - tW_1, t \in [0, 1]$ (Bridge)
(iii) $X_t := (1 - t)W(t/(1 - t)), t \in [0, 1[$ (Mystery!)

Ex. 7:
A fractional Brownian motion (fBM) with exponent H is defined as a Gaussian process with zero mean and correlation function:

$$\langle W_t^H W_s^H \rangle = \frac{1}{2}\left(s^{2H} + t^{2H} - |t - s|^{2H}\right) \tag{3}$$

i) Is the fBm stationnary ? What about its increments $\Delta W_\tau := W(t + \tau) - W(t)$?
ii) Show that H=1/2 corresponds to canonical BM
iii) Show the scaling symmetry $(W_{\alpha t}^H, t \geq 0) \overset{law}{\sim} (\alpha^H W_t^H, t \geq 0)$
iv) Deduce the scaling exponents of fBM, that is the exponents ξ_p such that

$$\xi_p = \lim_{\tau \to 0} \log \left\langle (\delta W_\tau^H)^p \right\rangle / \log \tau \tag{4}$$

v) Use Kolmogorov's regularity theorems to justify the existence of a continuous version of fBM and study Hölder differentiability of fBM.

Ex. 8: Simulation of Gaussian processes using Cholesky decomposition
Let $\mathbf{C} \in \mathbb{R}^{d \times d}$, positive definite symmetric matrix.

(o) Recall/observe/compute that the characteristic function of a gaussian random variable with covariance matrix $\boldsymbol{C} \in \mathbb{R}^{d \times d}$ and mean $\boldsymbol{\mu}$ is $\phi(\boldsymbol{t}) := \left\langle e^{i\boldsymbol{t}\cdot\mathbf{X}} \right\rangle = e^{i\boldsymbol{\mu}\cdot\boldsymbol{t} - (1/2)\boldsymbol{t}^T\boldsymbol{C}\boldsymbol{t}}$

(i) Show that there exists $L \in \mathbb{R}^{d \times d}$, lower diagonal such that $LL^T = \Gamma$ (Hint: Direct calculation and induction)

(ii) Let $\mathbf{X}(\varpi) \in \mathbb{R}^n$ a random normal vector with iid entries $X_i \sim \mathcal{N}(0, 1)$. Define the random variable $Y := LX$. Is Y Gaussian ?

(iii) Compute the average $\langle Y \rangle$ and the covariance matrix $C_{ij} := \langle (Y - \langle Y \rangle)_i (Y - \langle Y \rangle)_j \rangle$

(iv) Simulate your favorite Gaussian process using Cholesky decomposition using your favorite programming language. Illustrate the Gaussian behaviour of the pdf, and retrieve the hallmark signatures of those processes using Monte-Carlo averaging.