Brownian Motion

① As a diffusion [Motivational argument].

② As a Gaussian process. [Formal Definition].

③ Ensemble property of B.M.
→ Joint PDF
→ Symmetry.
→ Markov

④ Path Properties of B.M.
→ Hölder
→ Nowhere Differentiability.

Add-ons

Ⓐ Weak Stationarity & Taylor Diffusion
(why BM matters!)

Ⓑ Gaussian processes. (Exercises)

Sources

Pavliotis,
Evans, Lecture note.
Frisch, "Legacy".
Oksendal.

(BM) .

① B.M as a diffusion process

(A) Observations

* B.M ties the phenomenon of macroscopic diffusivity to a concept of "microscopic randomness".

* B.M is both a Physical Phenomenon and a math construction.

- Name comes from Robert Brown, an early 19th century botanist interested in Fertilisation of plants through pollen dissemination. He observed that inanimate suspensions of small pollen particle showed non-stop "zig-zaging" motion and never came to rest.

$\Longrightarrow$ Zigzaging motion related to self-propelling mechanism? ["Life"?].

* Link between Diffusive motion & microscopic molecular motion was made by Einstein (PhD + 1905 paper)

$$D = \frac{k_B T}{\gamma}, \quad \gamma \to 6\pi\eta a$$ provides a way to estimate Avogadro's number

* Perrin: systematic experiments to get $N \sim 6 \cdot 10^{23}$.

[confirming Einstein's own experiments involving sugar in water!].

→ Full account [Fourier, Fick, Delsaux, Langevin, Smoluchowski ...].

See Subtle is the Lord, by A. Pais.

$1.38 \cdot 10^{-23}$

* Estimate: "Diffusive power of water" $\rho = 1 kg/m^3$ $\eta = 10^{-6} m^2 . s^{-1}$, $k_B = \frac{R}{N}$

$$\sqrt{D\delta}/a \sim \frac{1}{a^{3/2}} \longrightarrow 0 \quad a \gg 1$$
$$\longrightarrow \infty \quad a \ll 1.$$

See also notebook "Brownian Motion Illustrated".

$\delta = 1 h$

$a = 10^{-9} m$ (water mol) $\Rightarrow \sqrt{D\delta} \sim 10^{6} a$

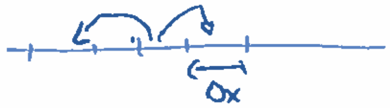
$a = 10^{-6} m \Rightarrow \sqrt{D\delta} \sim 28 a$

$a = 10^{-3} m \Rightarrow \sqrt{D\delta} \sim 10^{-3} m$.

$a = 1 \Rightarrow \sqrt{D\delta} \sim 3 \cdot 10^{-7} m$.

(B) Extracting diffusion from simple (RW)   [Heuristics].

* $\underline{1D}$ RW with "jump transition probability" $\phi_\tau[\Delta]\,d\Delta$



Proba to perform a jump Δ within τ.

* Define $n(x,t)\,\Delta x \doteq N\,c(x,t)\,\Delta x$ = number of part $\in [x, x+\Delta x]$.

* Assume $\langle \Delta \rangle = 0$, $\langle \Delta^2 \rangle < +\infty$, $\int n(x,t)\,dx = cste \doteq N$.

* The Master equation for c is:

$$c[x, t+\tau] = \int d\Delta\, \phi[\Delta]\, c[x-\Delta, t].$$

$$\simeq c + \tau\,\partial_\tau c \qquad \simeq c - \langle \Delta \rangle c_x + \frac{\Delta^2}{2} c_{xx}$$

$\langle \Delta \rangle \to 0$

$\Rightarrow$ A diffusion equation pops out!

Ex of "Diffusive scal

$$(n) \quad \boxed{\partial_\tau c = D\,\partial^2_{xx} c. \quad \underline{\text{with}} \quad D \doteq \frac{\Delta^2}{2\tau}}$$

conventional.

* From (n) one computes:

$$\langle x^0 \rangle = \int dx\, c(x,t) = 1,$$

$$\frac{d}{dt}\langle x^p \rangle = D \int x^p\, \partial^2_{xx} c = D\,p(p-1)\,\langle x^{p-2} \rangle.$$

$$\Longrightarrow \boxed{\begin{array}{l} \langle x \rangle = 0 \\ \langle x^2 \rangle = 2Dt \quad [+\langle x_0^2 \rangle] \end{array}}$$

How does "D" relate to "k_B"? Well this is left for later.

(C) Statistics from Law of Large numbers [Central Limit].

We now present a motivational [but rigorous] argument for the later coming definition of RW as a gaussian process

- Define the RV: "number of steps on the right side" (Δx)

$$S_n = \sum_{i=0}^{N-1} X_i \quad \text{with} \quad X_i \text{ iid} \quad P_X(x) = \frac{1}{2}[\delta(x) + \delta(x-1)]$$

$$\begin{cases} \langle X_i \rangle = \frac{1}{2} & \mu_X = \frac{1}{2} \\ \langle X_i^2 \rangle = \frac{1}{2} & \sigma_X^2 = \frac{1}{4} \end{cases}$$

- $R(t) \doteq S_N \Delta X + (n - S_n)(-\Delta X) \implies R_N = [2 S_n - n] \Delta X$
[Displacement after "n steps"].

$\implies$ Question: Stat of $R(T)$?

- Assume again a diffusive scaling, with $\Delta t = \frac{T}{N}$ [T fixed]
and assume $\exists D = \lim_{n \to \infty} \frac{\Delta X^2}{2 \Delta t}$.

then
$\langle R(T) \rangle = 0$.

$$\langle R^2(T) \rangle = [4 \langle S_n^2 \rangle - 4n \langle S_n \rangle + n^2] \Delta X^2 = n \Delta X^2$$

($\langle S_n \rangle$ = np.)

$\hookrightarrow n\langle X^2\rangle + n(n-1)p^2$

$$= 4n \langle X^2 \rangle + 4n(n-1)p^2 - 4n^2 p + n^2.$$

$$= 4n \frac{1}{2} + 4n(n-1)\frac{1}{4} - 4n^2 \frac{1}{2} + n^2 = 2n - n + \cancel{2n^2 - n^2} = $$

$$\implies \boxed{\langle R^2(T) \rangle = T \frac{\Delta X^2}{\Delta t} = 2DT}$$

- More precisely:

$$R(T) = 2\left(S_n - \frac{n}{2}\right) \Delta X \times \frac{\sqrt{n/4}}{\sqrt{n/4}} = \frac{\cancel{2}\left(S_n - \frac{n}{2}\right)}{\sqrt{n/4}} \frac{\boxed{\Delta x \sqrt{n}}}{\cancel{2}} = \sqrt{2DT}.$$

$$\implies \frac{R(T)}{\sqrt{2DT}} = \frac{S_n - \frac{n}{2}}{\sqrt{n/4}} = \frac{S_n - n\mu_X}{\sqrt{n \sigma_X^2}} \xrightarrow[\text{Law}]{} \mathcal{N}(0,1)$$

$$\implies P[R < x_0] = P\left[\frac{R}{\sqrt{2DT}} < \frac{x_0}{\sqrt{2DT}}\right] = \frac{1}{\sqrt{2\pi}} \int^{\frac{x_0}{\sqrt{2DT}}} \exp -\frac{t^2}{2} \, dt = \frac{1}{\sqrt{4\pi DT}} \int_{-\infty}^{x_0} \exp -\frac{t^2}{4D \dots}$$

| the limit process has Gaussian PDF! It is the limit obtained by this motivates the following definition |

② B.M as a stochastic Gaussian process.

Let us recall the following two definitions.

Def [Stochastic process]:

A stochastic process is a parametrized collection of random variables.

$\{X_t\}_{t\in T}$ $\forall t$ $X_t : \Omega \to \mathbb{R}^n$. defined on a proba. space (Ω, P)

Terminology: $\begin{cases} \omega \mapsto X_t(\omega) \text{ is a R.V } (\forall t) \\ t \mapsto X_t(\omega) \text{ is a path.} \end{cases}$ ω: "experiment / realization"

~~[illegible]~~

Def [Gaussian process]: A Gaussian process $\{X_t\}_{t\in T}$ is a stochastic process.

such that:

$\forall k \in \mathbb{N}$ $\forall t_0 < t_1 \dots < t_k \in T^k$

$$(X(t_0) - X(t_k)) \overset{\text{Law}}{\sim} \mathcal{N}(\mu_k, C_k)$$

for some $\mu_k \in \mathbb{R}^n$ $C_k \in \mathbb{R}^{n\times n}$ $C_k = C_k^T$ positive definite.

Explicitly: if $C_k = \Gamma_k \Gamma_k^T$ $(X_0 - X_k) \overset{\text{Law}}{\sim} \mu_k + \Gamma_k \mathcal{N}(0, I)$

Recall: • $X \sim \mathcal{N}(\mu, C) \iff P(X \in DX) = \dfrac{DX}{(\det C)^{1/2}(2\pi)^{n/2}} \exp -\frac{1}{2}(X-\mu)^T C^{-1}(X-\mu)$

$\mu \in \mathbb{R}^n$, $C \in \mathbb{R}^{n\times n}$, sym def ≥ 0.

• $n=1$ $X \sim \mathcal{N}(0,1) \iff P(X \in DX) = \dfrac{1}{\sqrt{2\pi}} \exp -\frac{1}{2} x^2 dx$

Rmk: • C is the correlation matrix: one checks that. if $\langle X \rangle = 0$.

$\langle X(t_0) X(t_1) \rangle = C_{t_0 t_1}$, $\langle (X_0 - \mu_0)(X_1 - \mu_1) \rangle = C_{01}$.

• (C, μ) entirely prescribes the moments of X cf Exo ~~sheet~~ on Gaussian pro[cess]

We can now formally define B.M. as:

Def 1 [B.M]:

A stochastic process $\{(W_t), t \in T\}$ that takes value in $\mathbb{R}$ is a (1D) B.M **iff** it satisfies the following properties:

(i) $W(0) = 0$ a.s.
(ii) ~~[illegible]~~ W is a gaussian process.
(iii) $\langle W_t W_s \rangle = \min(s,t)$ & $\langle W_t \rangle = 0$ $\forall s,t \in T$.
(iv) $t \mapsto W(t,\omega)$ is C^0 a.s.

We give right away an equivalent definition:

Def 2 [B.M]: $\{W_t, t \in T\}$ is B.M **iff**

(i) $W(0) = 0$ a.s.
[gaussian] (ii) $\forall t > s$ $W(t) - W(s) \overset{Law}{\sim} \mathcal{N}(0,1)\sqrt{t-s}$.
[indep] (iii) $\forall$ ~~0~~ $t_0 < t_1 < \dots < t_n \in T$, the events $\Delta W_i \doteq W(t_{i+1}) - W(t_i)$. $i \in [\![0; n-1]\!]$ are independent.
(iv) $t \mapsto W(t,\omega)$ is C^0 a.s.

1/ See Exo ✓

Thm (Wiener): B.M exist!

2/ We admit the thm [see Oksendal for details].

Essentially,

- (1) + (2) + (3) guarantees the existence of a gaussian s.p Y ~~whose finite dimensional distrib~~ ~~are gaussian~~: $(Y_{t_0} - Y_{t_n}) \sim \mathcal{N}(0, C)$ $C_{ij} = \min(t_i, t_j)$.

This is a consequence of the so-called Kolmogorov's "Consistency thm".

- Another thm due to Kolmogorov guarantees the existence of a s.p $\{W_t, t \in T\}$ whose finite-dimensional distributions are ~~[illegible]~~ a.s [illegible] a.s continuous. W_t is the canonical B.M

5'

- B.1 moments :

$$\boxed{\begin{aligned} \langle W_t^{2k+1} \rangle &= 0 . \\ \langle W_t^{2k} \rangle &= \frac{(2k)!}{2^k k!} t^k . \end{aligned}}$$

2/ <u>Exercice</u>

① <u>Compute</u>: $\langle e^{itW} \rangle = e^{-t^2/2}$

② <u>Taylor</u>: $\sum_k i^k t^k \frac{\langle W^k \rangle}{k!} \stackrel{?}{=} \sum_k (-1)^k \frac{t^{2k}}{2^k k!}$

- <u>Rmq</u> [Wick's thm]. / gaussian Integration by part [<u>see Frisch / Chapter 3</u>].

(i) For a random variable $X \sim \mathcal{N}(0, C)$ $X(\omega) \in \mathbb{R}^n$:

$$\begin{aligned} \langle X_i f(X) \rangle &= \sum_j \langle X_i X_j \rangle \left\langle \frac{\partial f}{\partial X_j} \right\rangle \\ &= \sum_j C_{ij} \left\langle \frac{\partial f}{\partial X_j} \right\rangle \end{aligned}$$

(ii) <u>In particular</u>:

$$\langle X_i \rangle = 0 .$$

$$\langle X_i X_j \rangle = C_{ij}$$

$$\langle X_i X_j X_k X_\ell \rangle = C_{ij} C_{k\ell} + C_{ik} C_{j\ell} + C_{i\ell} C_{jk}$$

(iii) <u>More generally</u>

$$\langle X_{i_1} \dots X_{i_{2m+1}} \rangle = 0 .$$

$$\langle X_{i_1} \dots X_{i_{2m}} \rangle = \sum \langle X_{i_{\ell_1}} X_{i_{\ell_2}} \rangle \cdots \langle X_{i_{\ell_{2m-1}}} X_{i_{\ell_{2m}}} \rangle .$$

$\forall m \quad i_1 \dots i_{2m}$.

where $(\ell_1, \ell_2) \dots (\ell_{2m-1}, \ell_{2m})$ is an arbitrary partition of $(1 \dots 2m)$ in pairs.

The summation is over all partitions.

$\#$ <u>partitions</u> $= (2m-1) \times (2m-3) \dots \times 1$.

$\#_m = (2m-1) \times \#_{m-2}$

Appendix :

Kolmogorov's continuity thm :

Let $(X_t, t\in T)$ a stochastic process such that:

$\forall (t,s) \in T^2$ $\boxed{\exists\, \alpha, \beta, D}$ $\langle |X_t - X_s|^\alpha \rangle \leq D|t-s|^{1+\beta}$

Then there exists a C^0 version of X.

Def : [Version of a s.p].

Let $(X_t, t\in T)$, $(Y_t, t\in T)$ be s.p on (Ω, P) .

X_t is a version of Y_t iff $P\{\omega : X_t(\omega) = Y_t(\omega)\} = 1$ $\forall t\in T$

ie $\forall t\in T$ $X_t = Y_t$ a.s.

Rmk : In that case, Y_t & X_t have same finite dimensional distributions.

They are similar in law but their path properties differ.

Application :

B.M : [Take $\alpha = 4$, $\beta = 1$, $D = 3$ $\Rightarrow$ B.M has a C^0 version.

$\mathbb{E}\langle W_t^4 \rangle = \frac{4!}{2^2\, 2!} t^2 = \frac{3.4}{4} t^2$] .

Kolmogorov's thm of consistency.

Let $\{\mu_{t_1 \dots t_n}\}_{\{t_i\}}$ a family of marginals st that :

$\forall k \in \mathbb{N}$ $\forall t_1 \dots t_n$ $\forall \sigma$ $\mu_{t_1 \dots t_n}[F_1 \dots F_n] = \mu_{t_{\sigma(1)} \dots t_{\sigma(n)}}[F_{\sigma(1)} \dots F_{\sigma(n)}]$

$\mu_{t_1 \dots t_n}[F_1 \dots F_n] = \mu_{t_1 \dots t_n t_{n+1} \dots t_{n+m}}[F_1 \times \dots F_n \times \mathbb{R} \times \dots \mathbb{R}]$

then $\exists (\Omega, P)$, $X_t : \Omega \to \mathbb{R}$ such that .

$\mu_{t_1 \dots t_n}(F_1 \dots F_n) = P[\forall i \in [\![1;n]\!]\ X_{t_i} \in F_i]$ for $\{t_i\}, F_i$.

③ Ensemble properties of B.M.

Ⓐ PDF

Thm [Averages]:

For all $n > 0$, $0 = t_0 < t_1 \dots < t_n$, $\varphi : \mathbb{R}^n \to \mathbb{R}$ observable.

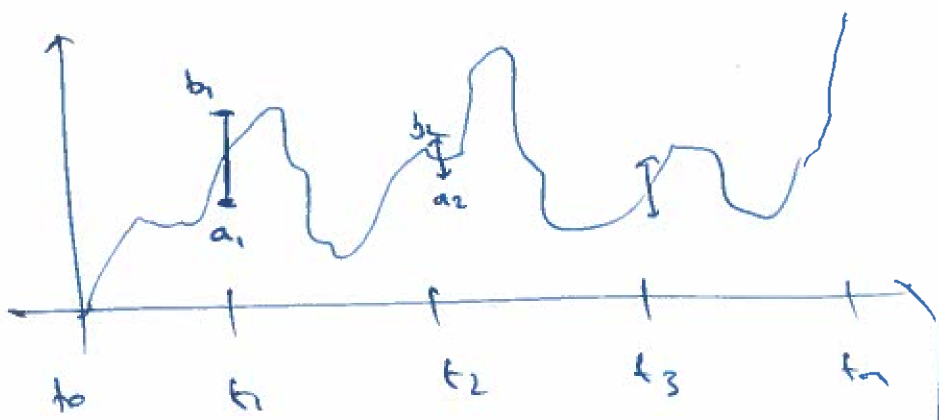
$$\langle \varphi[W(t_1) \dots W(t_n)] \rangle = \int dx_1 \dots dx_n\, \varphi(x_1 \dots x_n)\, g(x_1, \tau_1 | x_0) \dots g(x_n, \tau_n | x_{n-1})$$

where
$$\begin{cases} \tau_i = t_i - t_{i-1} \quad [\, i \in [\![1; n]\!] \,]. \\ g(x_i, \tau_i | x_{i-1}) \doteq \dfrac{1}{\sqrt{2\pi\tau_i}} \exp - \dfrac{\Delta x_i^2}{2\tau_i} = g(\Delta x_i, \tau_i | 0) \\ \Delta x_i = x_i - x_{i-1}. \end{cases}$$

D/ (a-f 2)
$$\langle \varphi(W_1 \dots W_n) \rangle = \int dx_1 \dots dx_n\, \varphi(x_1 \dots x_n)\, P(x_1 \dots x_n) = \int d\Delta x_1 \dots d\Delta x_n\, \rho(\Delta x_1 \dots \Delta x_n)\, \tilde{\varphi}$$
$$= \int \Big[\prod d\Delta x_i\, \rho(\Delta x_i)\Big]\, \rho(\Delta x_1 \dots \Delta x_n)$$
indep. incr.
$$= \int \prod_i g[\Delta x_i, \tau_i | 0]\, \rho = \int dx_1 \dots dx_n \Big[\prod_i g_{\tau_i}(x_i | x_{i-1})\Big] \varphi.$$

Thm: [Joint density of BM]

$$P[x_1 \dots x_n] = \frac{1}{Z} \exp - \frac{1}{2} \sum_{i=1}^{n} \Delta x_i^2 / \tau_i . \quad (*)$$
$$Z = (2\pi)^{n/2} \Big[\prod_{i=1}^{n} \tau_i\Big]^{1/2}.$$



- $P[\forall i\; W(t_i) \in [a_i; b_i]]$
$$= \int dx_1 \dots dx_n \prod_i \Theta(x_i - a_i)\Theta(b_i - x_i)\, p(x_1 \dots x_n),$$

- Formally (*) can be thought as a path integral:

Let $x \in C^1[0,T] \cap L^2([0,T])$.

$$P[x]\mathcal{D}x \sim \exp - \frac{1}{2} \int_0^T \dot{x}^2 dt$$

(B) Symmetries.

Thm The following transformed B.M are equivalent in law to canonical (BM)

- [Inversion] $(-W_t,\ t\geq 0)$
- [Rescaling] $\left(\frac{1}{\sqrt{\alpha}} W_{\alpha t},\ t\geq 0\right)$ $\alpha>0$.
- [Time Reversal] $(W_{1-t}-W_1,\ t\in[0,1])$.
- [Time Inversion] $(tW_{1/t},\ t\geq 0)$

D/ Use def (i).
- gaussianity: comes from known X gaussian $\Rightarrow X_{f(t)}$ gaussian for $f\nearrow$ or $f\searrow$.
- $\langle \tilde W_t \tilde W_s\rangle \underset{\uparrow \text{Ex}}{=} \frac{1}{\alpha}\langle W_{\alpha t} W_{\alpha s}\rangle = \frac{\alpha}{\alpha}(s\wedge t)$ ✓.
- C^0 ok
- $W(0)=0$ a.s. ok for last time then comes from, $\frac{W_t}{t} \xrightarrow[a.s.]{} 0 \quad t\to\infty$. $[W=\sum \Delta W_i$ (4) law of large numbers

Rmk The scaling ppty means that BM is self-similar and has "monofractal scaling"

$$\langle \Delta W_\tau^p\rangle = \langle (W_\tau - W_0)^p\rangle \doteq \langle (W_{s+\tau}-W_s)^p\rangle$$
$$\underset{\uparrow\ \text{"scaling trick"}}{=} \langle \sqrt{\tau}^{\,p}(W_{\tau/\tau}-W_{0/\tau})^p\rangle$$
$$= \tau^{p/2}\langle \Delta W_1^p\rangle.$$

ie $\Delta W_\tau^p \sim \tau^{\zeta_p}$ with $\zeta_p = p/2$.

We retrieve the results $\langle \Delta W_\tau^p\rangle = \langle W_\tau^p\rangle = 0$ if $p=2n+1$ (C)
$= \frac{(2n)!}{n!\,2^n}\,\tau^n$ if $p=2n$

obtained previously by direct computation.

(4) Path properties of B.M.

- In Φ., or quite often, one sees the notation

$$\text{"} \; y(t) \doteq \frac{dW}{dt} \; \text{"}$$

as a defining identity for white noise.

- This may look shocking! $\langle \Delta W_\delta^2 \rangle = \delta$ hints that BM is irregular!

  (we will see that this is indeed the case in a pathwise sense)

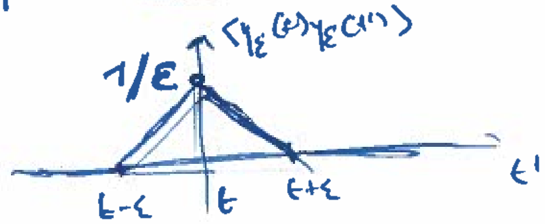
- Let us fix $\varepsilon > 0$ and write:

$$y_\varepsilon(t) \doteq \frac{\Delta W_\varepsilon(t)}{\varepsilon} = \frac{W(t+\varepsilon) - W(t)}{\varepsilon}.$$

Then $y_\varepsilon(t)$ is gaussian.

y_ε is stationary $\langle y_\varepsilon(t) y_\varepsilon(t') \rangle = \frac{\langle \Delta W_\varepsilon(t) \Delta W_\varepsilon(t') \rangle}{\varepsilon^2} \doteq \mathcal{E}_\varepsilon(\tau$

$$= \frac{2(t \wedge t') + \varepsilon - (t+\varepsilon) \wedge t' - t \wedge (t'+\varepsilon)}{\varepsilon^2}.$$



$$= \begin{cases} 2\,[\text{illegible}] + \varepsilon - t' \ldots & t' < t - \varepsilon \\ t' + \varepsilon - t' \ldots & t - \varepsilon < t' < t \\ t' - t + \varepsilon. & \\ & t' = \end{cases}$$

$\mathcal{E}_\varepsilon(0) \xrightarrow[\varepsilon \to 0]{} \infty$,

$\int \mathcal{E}_\varepsilon(\tau)\, dt = 1$

$\Rightarrow$ " $\mathcal{E}_\varepsilon \to \delta$ " [Dirac].

- The power spectrum of :

$y_\varepsilon = \lim_{\varepsilon \to 0} y_\varepsilon$ is :

$$s[\omega] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t}\, \delta[t]\, dt = \frac{1}{2\pi}.$$

Every frequence contributes equally $\Rightarrow$ the noise is white!

(later, we will see examples of red noise!)

[see Course on stationary process

c) Markov property of BM

Consider the joint probability for 2 points at $t_1 < t_2$

$$P_2\left[X(t_1)\in\Delta x_1 \;\&\; X(t_2)\in\Delta x_2\right] \doteq \Delta x_1 \Delta x_2 \; p_1(x_1,t_1|0)\; p_2(x_2,t_2|x_1,t_1)$$

$$= \Delta x_1 \Delta x_2 \; p_1(x_1,t_1)\; p_1(x_2, t_2-t_1) \qquad (\text{Property of } p_1)$$

$\Rightarrow$ In particular,

$$P_1\left(X(t_2)\in\Delta x_2\right) = \int_{x_1} P_2\left[X(t_1)\in dx_1 \;\&\; X(t_2)\in\Delta x_2\right]$$

$$= \Delta x_2 \int p_1(x_1,t_1)\, p_1(x_2,t_2|x_1,t_1) \quad -(*)$$

Besides,

$$P_1\left(X(t_2)\in\Delta x_2\right) = \Delta x_2 \; p_1(x_2,t_2) \,. \qquad (**)$$

Equating $(*)$ & $(**)$ yields the

Chapman-Kolmogorov equation for the one-time density:

$$\boxed{\forall\, t_1 < t_2 : \quad p_1(x_2,t_2) = \int dx_1\, p_1(x_1,t_1)\, p_2(x_2,t_2|x_1,t_1)}$$

Heat equation from Chapman Kolmogorov

Consider: $t_2 = t+\tau$ & $t_1 = t$.

$$p_1(x_2,t+\tau) = \int p_1(x_1,t)\underbrace{p_1(x_2,t+\tau|x_1,t)}_{\substack{\shortparallel \\ p_1[x_2-x_1,\tau] = \chi_\tau(\Delta)}} dx_1 = \int d\Delta\, \chi_\tau(\Delta)\, p_1[x_2-\Delta, t$$

$\Rightarrow$ Taylor expansion: $\left[\partial_\tau \rho = \frac{1}{2}\partial^2_{xx}\rho = D\,\partial^2_{xx}\rho\right]$ with $D=\frac{1}{2}$ cf 2^{nd} ...

3

In fact, the ensemble non-regularity translate into a pathwise non-regularity:

Thm [Kolmogorov]:

Let X_t be a s.p with C^0 path a.s. such that:

$$\langle |X_t - X_s|^\beta \rangle \leq c|t-s|^{1+\alpha} \quad \text{for } \beta, \alpha > 0 \quad 0 \leq t, s.$$

then for each $\gamma \in]0, \frac{\alpha}{\beta}[$, $T > 0$ & a.e ω:

$$\exists K(\omega, \gamma, T), \quad |X(t,\omega) - X(s,\omega)| \leq K|t-s|^\gamma \quad \forall\, 0 \leq s, t \leq T$$

2/ See Evans.
Proof uses standard proba tools [Borel-Cantelli; Chebychev].
Starting point. $A_n \stackrel{\circ}{=} \left\{ \left| X\left(\frac{i+1}{2^n}\right) - X\left(\frac{i}{2^n}\right) \right| > \frac{1}{2^{n\gamma}} \text{ for } i \in [0, 2^n[\right.$

Evaluate $P[A_n] \leq C\, 2^{n[-\alpha+\gamma\beta]}$

$\Longrightarrow$ $P[A_n\ \text{i.o}] = 0.$

Application:

B.M use $\beta = 2m$, $\alpha = m-1$ $[\langle W_t^{2m} \rangle \leq c\, t^m]$

Hence for each $\gamma \in]0, \frac{1}{2} - \frac{1}{2m}]$ $t \mapsto X_t(\omega)$ is uniformly Hölder γ.

ie $\gamma \in]0, \frac{1}{2}[$.

In reality the bound is sharp, as stated by the following

<u>Thm</u>: (i) For each $\frac{1}{2} < \gamma \leq 1$ and almost every ω

$t \mapsto W(t,\omega)$ is nowhere Hölder with exponent γ.

(ii) For almost every ω, $t \mapsto W(t,\omega)$ is in particular nowhere differentiable.

2/ <u>See Evans</u>.

<u>Rmk</u>: Easy to show:

$\forall t_i - t_n \quad P[\forall i \, |\Delta W_i(t_i,\tau)| \geq \tau^\gamma \eta]$

$\doteq W(t_i+\tau) - W(t_i)$.

<u>Let</u> $\eta > 0$ & τ sufficiently small:

Then
$$P[\forall i \, |\Delta W_i| \geq \tau^\gamma \eta] = \prod_i P[|\Delta W_i| \geq \tau^\gamma \eta]$$
$$= 2 \prod_i \int_{\tau^\gamma \eta}^{\infty} \frac{1}{\sqrt{2\pi\tau}} \exp -\frac{x^2}{2\tau} \, dx.$$
$$\underset{u = \frac{x}{\sqrt{\tau}}}{=} 2 \prod_i \int_{\tau^{\gamma-\frac{1}{2}}\eta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \, du \xrightarrow[\tau \to 0]{} \Delta \quad \text{if } \gamma > \frac{1}{2}$$