Exercises related to Stochastic Integration

Ex. 1:
Show that the solutions of the SDE:

$$dX = -\frac{1}{2}e^{-2X}dt + e^{-X}dW \quad \text{with} \quad X(0) = x_0 \tag{1}$$

blow up at finite random time.

Hint: Study the process $Y := e^X$.

Ex. 2:
Let $\mathbf{W} = (W_1, W_2 \cdots W_d)$ be a d-dimensional Brownian motion.

Show that the stochastic process $R = \left(\sum_{i=1}^d (W_i)^2\right)^{1/2}$ has the stochastic differential:

$$dR = \sum_{i=1}^{d} \frac{W_i dW_i}{R} + \frac{d-1}{2R}dt \tag{2}$$

Hint: Multi-dimensional Ito's formula for $u(x1, x_2..., x_d) := (\sum_{i=1}^d x_i^2)^{1/2}$

Ex. 3:
Show that $X_t := (1-t)\int_0^t \frac{dW(s)}{1-s}$ solves

$$dX = \frac{-X}{1-t}dt + dW \quad \text{with} \quad X(0) = 0 \quad \text{for} \quad t \in [0,1] \tag{3}$$

Show that $X_t \overset{\text{law}}{=} W_t - tW_1$, that is X_t is a Brownian bridge.

Hint: Ito's formula for $Y := \dfrac{X}{1-t}$

Ex. 4: Exit times statistics from the backward equation.
Let X_t be a (1D) stochastic process with stochastic differential $dX_t = Fdt + GdW$. We assume that F and G are time-independent.

*(i)*Assuming that F and G are time-independent, show that the transition probabilities are invariant by time-translations: $p(x, t+\tau|x_0, t_0+\tau) = p(x, t|x_0, t_0)$

(ii) Let $x_0 \in]a; b[$ and

$$\tau(x_0) := \inf\{t > 0; X(t|x_0) \notin]a; b[\}$$

Suppose $p_a(x, t|x_0, t_0)$ solves the backward Kolmogorov equation with the non-entering conditions: $p_a(x, t|a, t_0) = p_a(x, t|b, t_0) = 0$. Argue that

$$\langle \tau(x_0)^p \rangle = p\int_0^\infty t^{p-1}N(t)dt \qquad \text{with} \qquad N(t) := \int_a^b dx p(x, t|x_0, 0)$$

(iii) Using formal caculations, show that $\langle \tau(x_0)^p \rangle$ solves:

$$\mathcal{L}\langle \tau(x_0)^p \rangle = -p\left\langle \tau(x_0)^{p-1}\right\rangle \quad \text{with boundary conditions} \quad \langle \tau(a)^p \rangle = \langle \tau(b)^p \rangle = 0.$$

Ex. 5: Kramer's problem.
Consider the (1D) (overdamped) Langevin dynamics in a double-well potential:

$$dX_t = -U'(X_t)dt + \sqrt{2\kappa}dW \text{ with } U(x) := \Delta(x^2-1)^2 \text{ and } \Delta > 0$$

(i) Suppose that $\kappa = 0$. Show that -1 and $+1$ are stable equilibria and 0 is an unstable equilibrium point.
(ii) Suppose now $\kappa > 0$. Show that $p_\beta \sim e^{-\beta U}$ is the equilibrium distribution, provided $\kappa = \beta^{-1}$.
(iii) Let $\tau(x_0) = \inf\{t > 0 : X(t|x_0) = 0\}$. Show that $\langle \tau(x) \rangle$ solves the PDE:

$$\beta^{-1} e^{\beta U} \partial_x \left(e^{-\beta U} \partial_x \phi \right) = -1 \text{ with boundary conditions } \phi'(-\infty) = \phi(0) = 0.$$

(iv) Show that in the limit $\beta \to \infty$, the average time a particle stays on a given well is $\tau \simeq \pi e^{\beta\Delta}/(2\Delta\sqrt{2})$.