Einstein relation from Langevin equation

Consider the following SDE, where x and v represent two real-valued stochastic processes.

$$dx = v\,dt \tag{1}$$

$$m\,dv = -\gamma v\,dt + \sqrt{2\kappa}\,dW \tag{2}$$

This SDE is a simplified instance of the Langevin equation. It describes the motion of a big particle with mass m and diameter a suspended in a fluid at rest and constantly colliding with a bath of smaller fluid particles, modeled by the noise term. The term $-\gamma v$ represents the Stokes drag, with the coefficient is explicitly $\gamma = 6\pi\eta a$, and η represents the dynamics viscosity of the solvent. Our aim here is to recover Einstein's Formula for the diffusivity from the analysis of this Langevin equation.

Step 1:
Show that the process:

$$v(t) = v_0 e^{-\gamma t/m} + \frac{\sqrt{2\kappa}}{m}\int_0^t dW(s) e^{-\gamma(t-s)/m}$$

solves the velocity equation with initial condition $v(t=0) = v_0$.

Hint: Use Itô's formula for the process $Y := e^{t\gamma/m} v$

Step 2:
Define the stationary velocity as :

$$v_\star(t) = \int_{-\infty}^t dW(s) e^{-\gamma(t-s)/m}.$$

Show that $v_\star$ indeed defines a stationary process with time correlation: $C(\tau) = \frac{\kappa}{m\gamma} e^{-\gamma|\tau|/m}$, and compute its correlation time τ_c.

Step 3:
Show that the Gibbs distribution

$$p_\beta = \sqrt{\frac{\beta m}{2\pi}} e^{-\beta m v^2/2}$$

is the equilibrium measure corresponding to the velocity process, provided $\beta := \gamma/\kappa$. Deduce the value $< v^2 >_\beta := \int_{\mathbb{R}} dv\, p_\beta(v)\, v^2$ at equilibrium.

Hint: Write the Fokker-Planck equation for the probability $p(v, t|v_0)$.

Step 4:
Consider the random variable $X(t, \varpi) := \int_0^t v_\star(s, \varpi) ds$ Show that

$$\frac{d}{dt} < X^2 >_t = 2D(1 - e^{-\gamma t/m}) \ \text{ with } \ D := 1/(\gamma\beta)$$

Study the asymptotics $t \gg \tau_c$ and deduce Einstein's formula. Observe that the result is independent of the mass of the particle. What about the asymptotics $t \ll \tau_c$?