Emergence of skewed non-Gaussian distributions of velocity increments in turbulence

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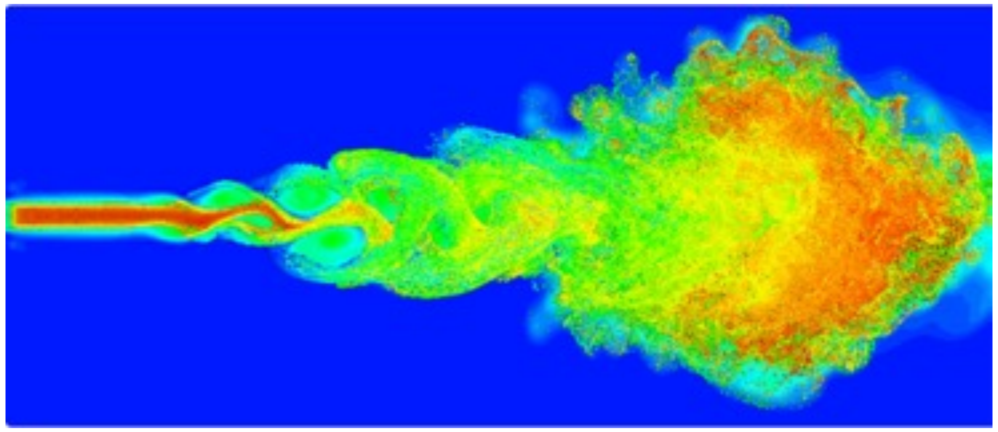
Outline

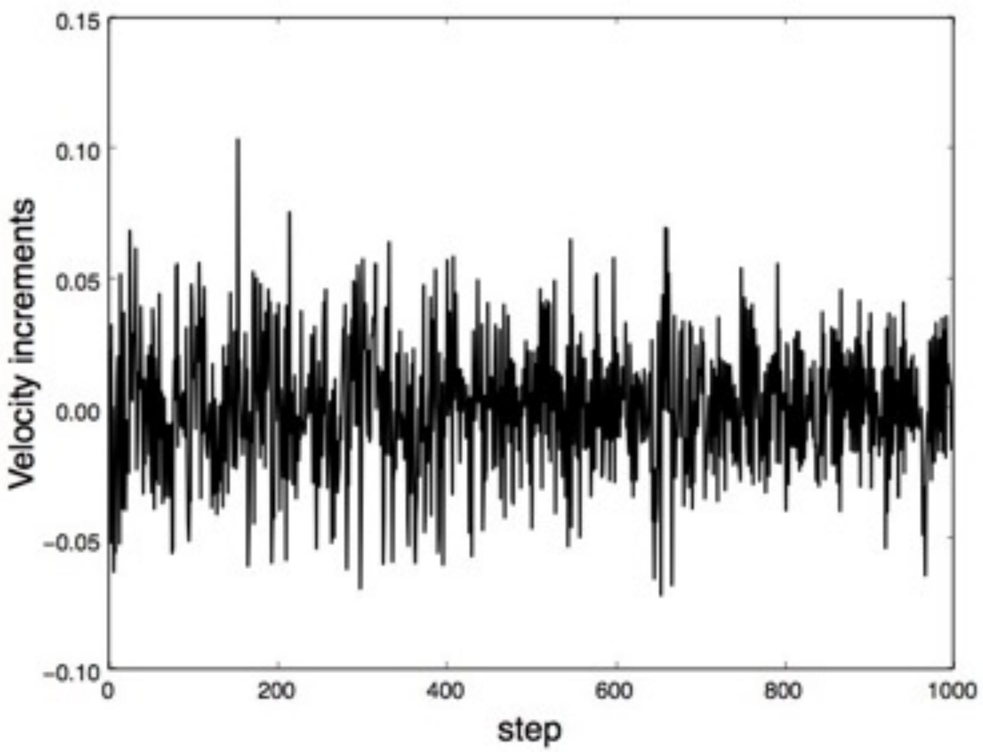
1. Introduction: Some Stylized Facts of Turbulence
2. Stochastic Hierarchical Models for Turbulence (H-Theory)
3. Comparison to DNS
4. Conclusions

Turbulence

- In a turbulent flow, **the fluid velocity fluctuates randomly** from point to point and from time to time.



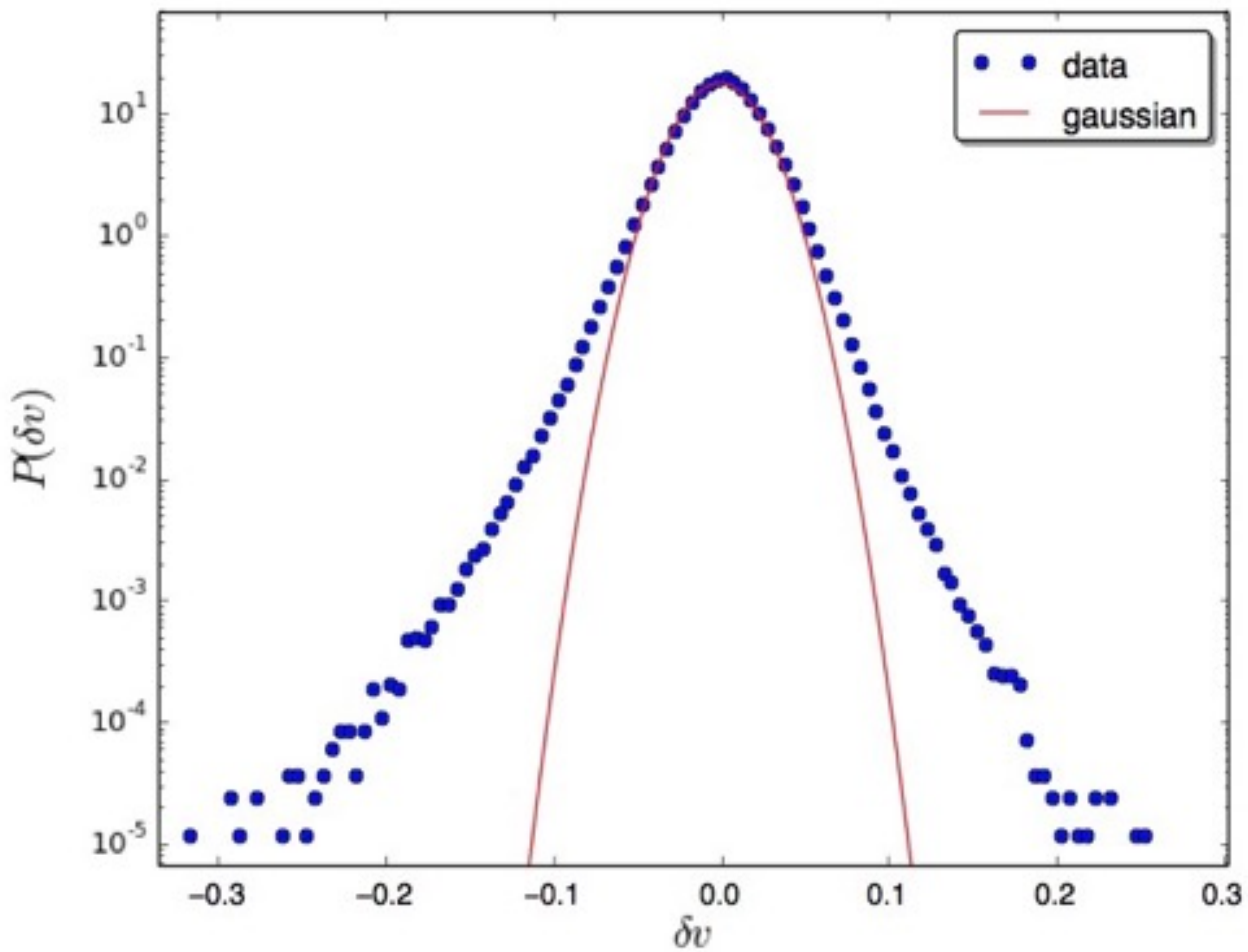
- **Longitudinal velocity Increments:** $\delta v_r = [\vec{v}(\vec{x}+\vec{r}) - \vec{v}(\vec{x})] \cdot \hat{r}$



What is the nature of the fluctuations? And why?

Turbulence: “Skewness”

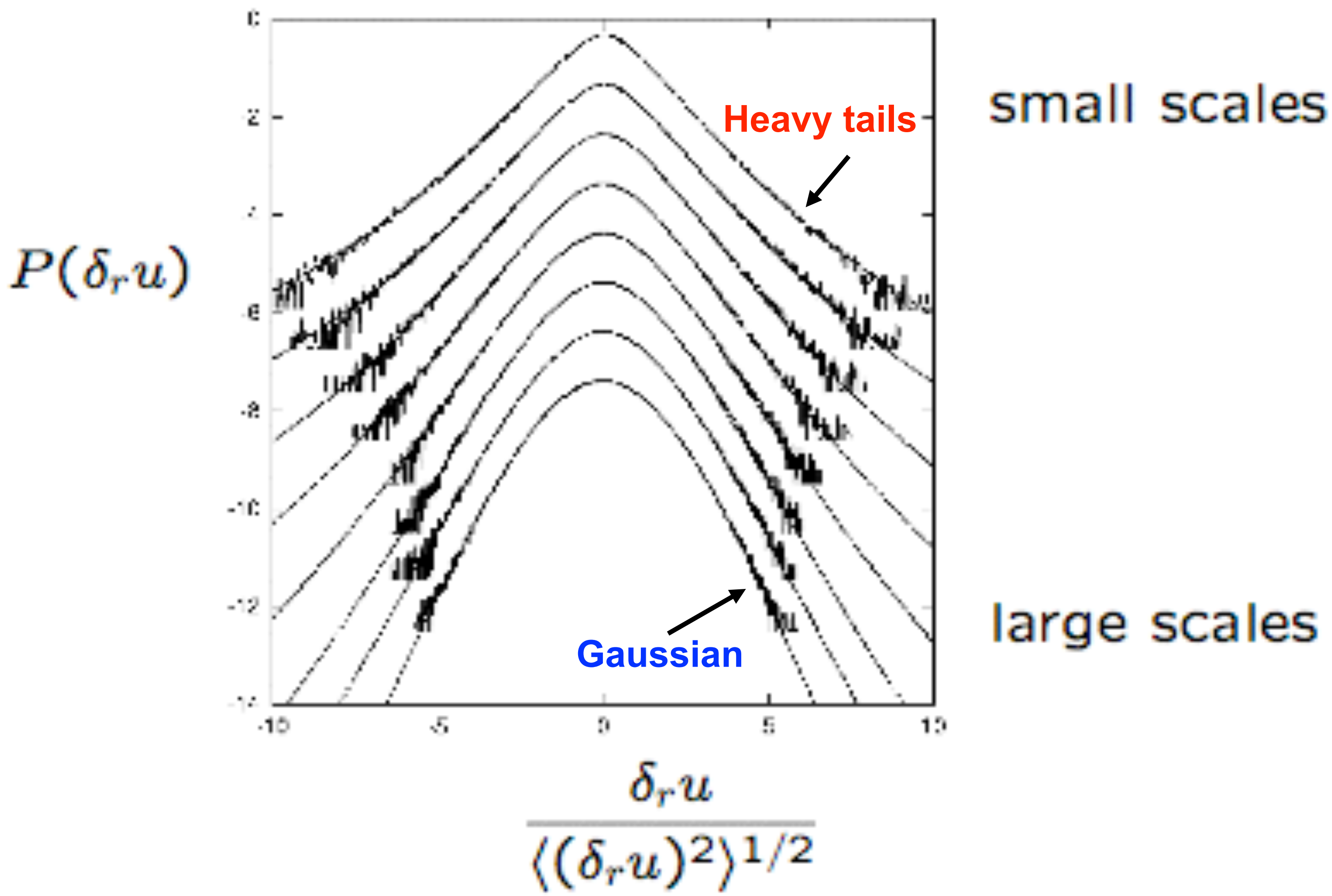
1. Asymmetric distributions at short scales.



- Kolmogorov’s 4/5 law : $\langle(\delta_r v)^3\rangle = -\frac{4}{5}\varepsilon r$ (negative skewness)
- **But why is that?**
- Typical “answers”: ‘time-reversal symmetry breaking’, ‘vortex stretching’.

Turbulence: “Intermittency”

2. Scale dependent distributions.

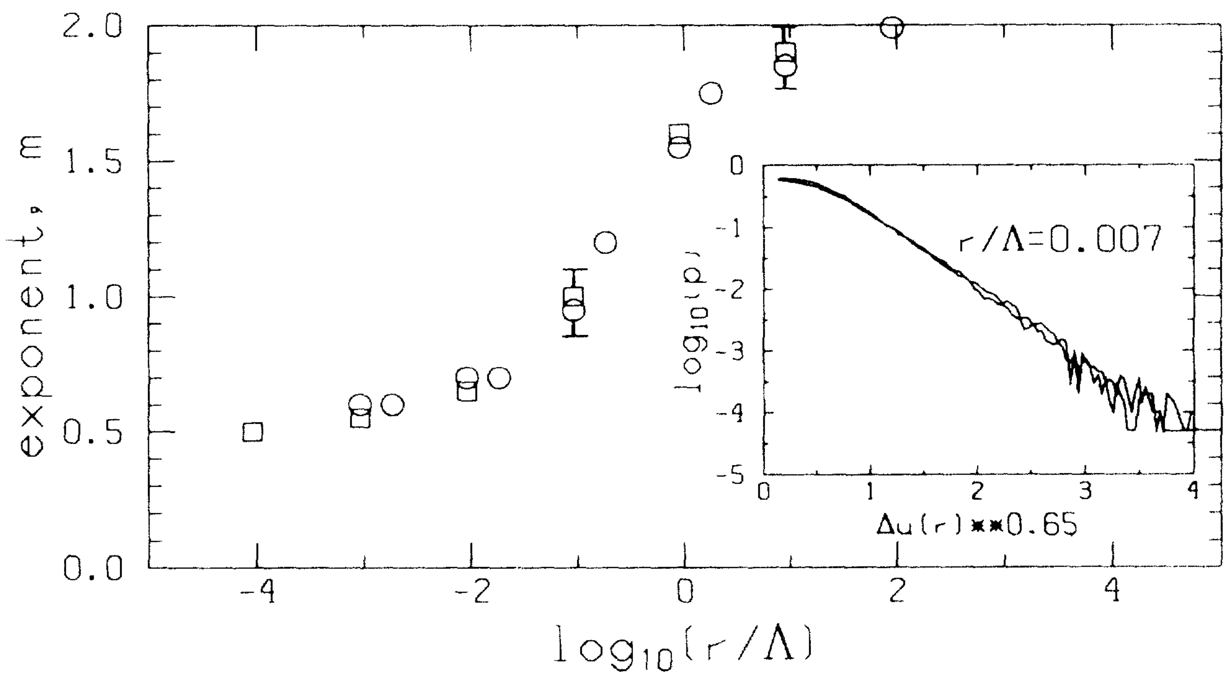


- Large-scale distributions are Gaussian.
- But short-scales are **NOT**?
- **Why is that?**

Turbulence: “Nature of Heavy Tails”

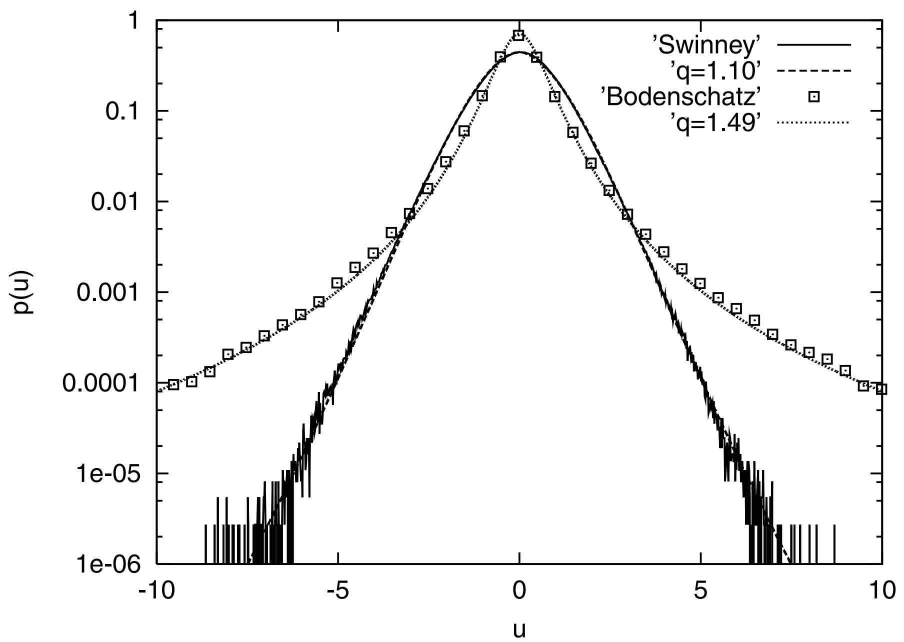
3. Tails decay slower than Gaussian

stretched exponential



or

power law



?

Kailasnath et al., PRL (1992),

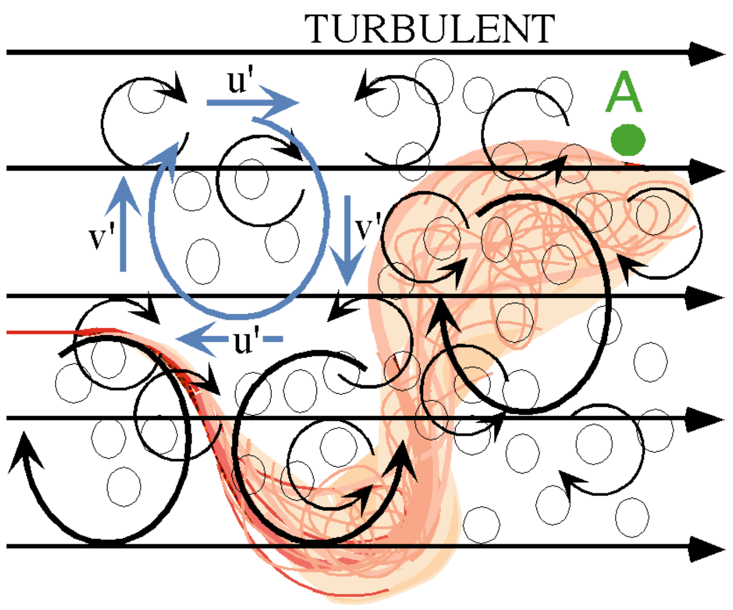
Beck, PRL (2001),

$$p(\delta v) \sim \exp\left[-a(\delta u)^m\right]$$

$$p(\delta v) \sim \frac{1}{\left[1 + a(q-1)(\delta u)^2\right]^{1/(q-1)}}$$

Origins of non-Gaussian Behaviour

- **Separation of time scales:** the particle velocity fluctuates much more rapidly than its environment (‘energy reservoir’).



- **Local quasi-equilibrium:** velocity increments during short times are Gaussian distributed.

$$p(\delta v_r|\varepsilon) = \frac{1}{\sqrt{2\pi\varepsilon}} \exp\left[-\frac{(\delta v_r)^2}{2\varepsilon}\right]$$

ε: rate of energy transfer

Compounding or Superstatistics

- **The marginal distribution is a mixture of Gaussians:**

$$p(\delta v) = \int_0^\infty P(\delta v_r|\varepsilon) f(\varepsilon) d\varepsilon = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{\varepsilon}} \exp\left[-\frac{(\delta v)^2}{2\varepsilon}\right] f(\varepsilon) d\varepsilon$$

- **Mixing distributions:**

i) Gamma distribution (Andrews et al., PoF, 1989).
ii) Log-normal distribution (Castaing et al., Physica D, 1990).
iii) Inverse-gamma distribution (Beck, PRL, 2001).

- **Drawbacks**:

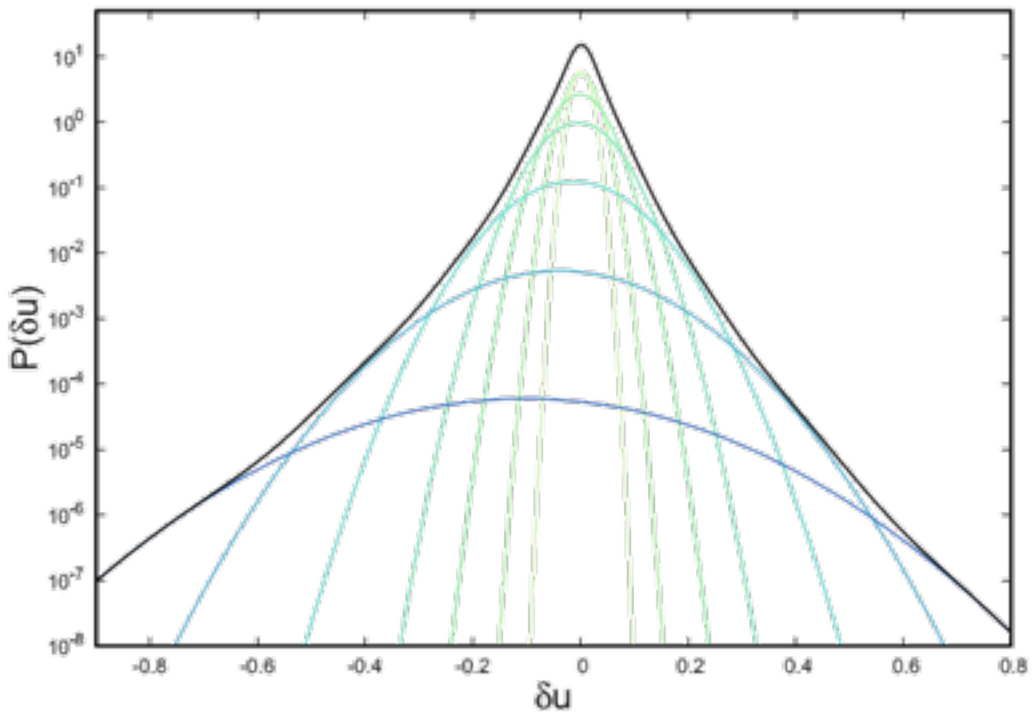
i) Adhoc choice of the mixing distribution.
ii) Symmetric distributions (zero skewness).

Origin of Skewness

- **Our Model:** Conditional distribution is a non-centered Gaussian

$$P(\delta v_r|\epsilon_r) = \frac{1}{\sqrt{2\pi\sigma^2_{\delta v_r|\epsilon_r}}} \exp\left[-\frac{(\delta v_r - \langle \delta v_r|\epsilon_r\rangle)^2}{2\sigma^2_{\delta v_r|\epsilon_r}}\right]$$

- **Superposition of Gaussian with nonzero mean** leads to non-Gaussian distributions:



$$P(\delta v_r) = \int_0^\infty P(\delta v_r|\epsilon_r) f(\epsilon_r) d\epsilon_r$$

Model Prescriptions for Mean and Variance

- The variance is a proxy measure for the energy flux:

$$\sigma^2_{\delta v_\ell|\varepsilon_\ell} = \varepsilon_\ell$$

- Mean-variance relation:

$$\langle \delta v_r | \varepsilon_\ell \rangle = \mu \left(\varepsilon_\ell - \langle \epsilon_\ell \rangle \right)$$

- This guarantees that $\langle \delta v_r \rangle = 0$, since

$$\langle \delta v_r \rangle = E[\langle \delta v_r | \varepsilon_\ell \rangle] = \mu \left(\langle \varepsilon_\ell \rangle - \langle \epsilon_\ell \rangle \right) = 0,$$

regardless of the choice of $f(\varepsilon)$.

Marginal Distribution of Velocity Increments

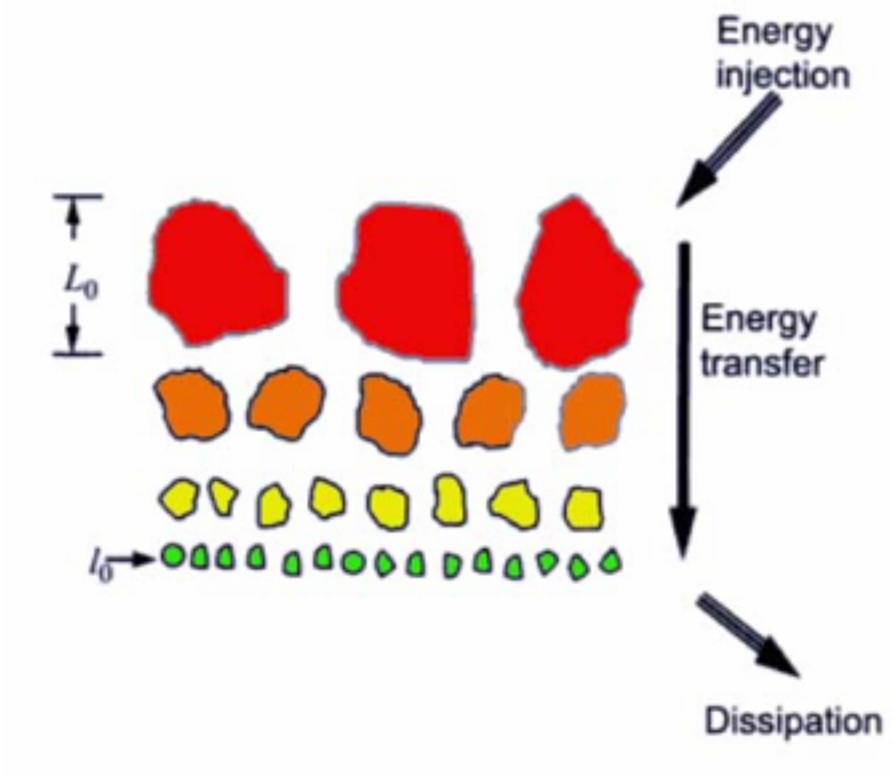
- The marginal distribution is then given by

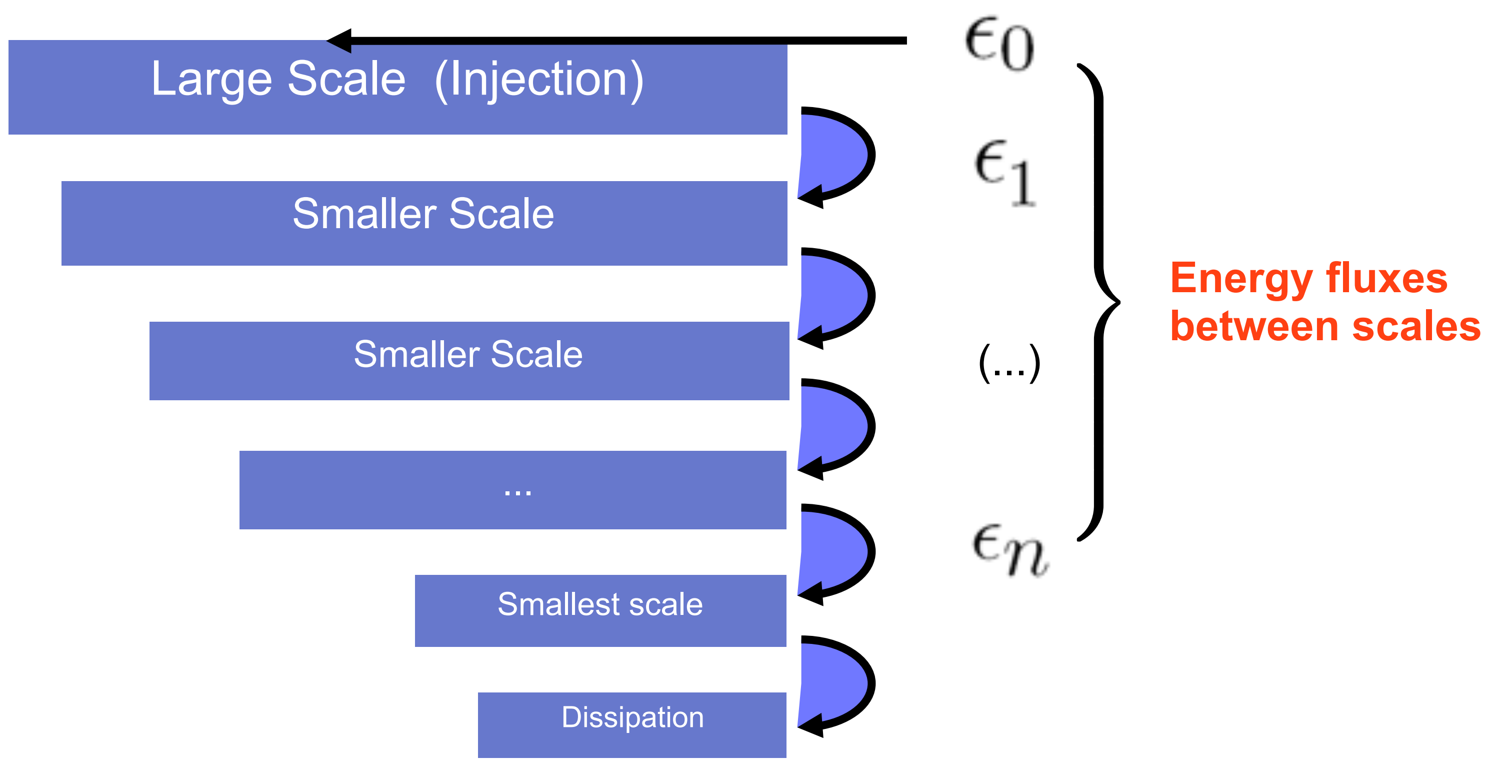
$$P(\delta v_r) = \int_0^\infty \frac{1}{\sqrt{2\pi\varepsilon_\ell}} \exp\left[-\frac{\left(\delta v_r - \mu(\varepsilon_\ell - \epsilon_0)\right)^2}{2\varepsilon_\ell}\right] f(\varepsilon_\ell) d\varepsilon_\ell$$

- **Separation of scales**: $\ell \gg r$, but ℓ is not known *a priori*—it must be determined from the velocity data!

- **Stochastic hierarchical model** for $f(\varepsilon_\ell)$: we introduced a general class of distributions that 'interpolates' between gamma and inverse-gamma.

Kolmogorov's Models of Energy Cascade






A. N. Kolmogorov

ε_n : rate of energy transfer at n-th level of the cascade

If ε_n is constant $\Rightarrow$ velocity fluctuations are Gaussian (K41)

Intermittency (K62): ε_n fluctuates with a log-normal distribution.

Generalized Hierarchical Model

- The hierarchical dynamical model is of the form

$$d\varepsilon_i = -\gamma_i \left(\varepsilon_i - \varepsilon_{i-1}\right) \left(1 + \alpha^2 \frac{\varepsilon_{i-1}}{\varepsilon_i}\right) dt + \kappa_i \sqrt{\varepsilon_i \varepsilon_{i-1}} dW_i,$$

  where

$$\frac{\kappa_i^2}{\gamma_i} = \beta$$

- The model has only two free parameters: α and β.
- Consider first $\alpha = 0$:

$$d\varepsilon_i = -\gamma_i \left(\varepsilon_i - \varepsilon_{i-1}\right) dt + \kappa_i \sqrt{\varepsilon_i \varepsilon_{i-1}} dW_i$$

- We then have **energy conservation** in the cascade:

$$\langle \varepsilon_i(t) \rangle = \varepsilon_0, \quad t \to \infty$$

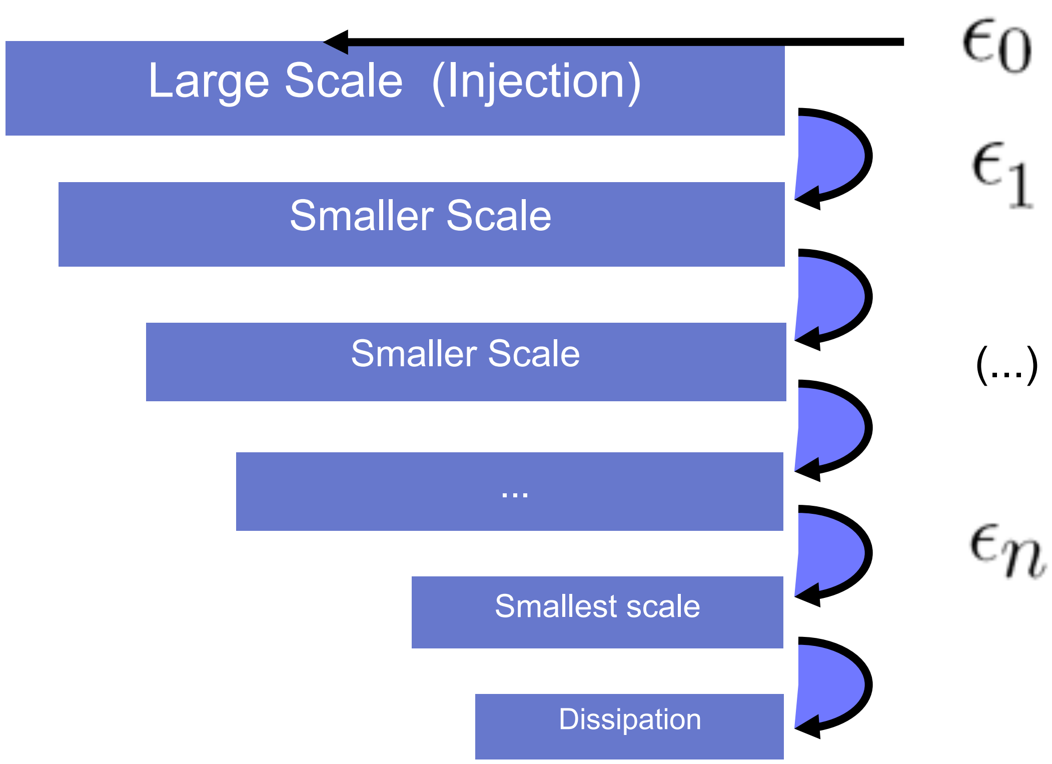
Generalized Hierarchical Model

- Suppose now $\alpha \neq 0$.

- In this case we can show that

$$\frac{\langle \varepsilon_i \rangle}{\langle \varepsilon_{i-1} \rangle} = 1 - \alpha^2 < 1, \qquad \text{as} \quad \alpha \to 0.$$

- Hence there is a **residual dissipation**, since $\langle \varepsilon_i \rangle < \langle \varepsilon_{i-1} \rangle$, meaning that **energy leaving scale** i **is less than the energy entering it.**



Generalized Inverse Gaussian

- Consider the full model:

$$d\varepsilon_i = -\gamma_i\left(\varepsilon_i - \varepsilon_{i-1}\right)\left(1 + \alpha^2\frac{\varepsilon_{i-1}}{\varepsilon_i}\right)dt + \kappa_i\sqrt{\varepsilon_i\varepsilon_{i-1}}dW_i,$$

- **Separation of time scale**: $\gamma_N \gg \gamma_{N-1} \gg ... \gg \gamma_1$.
- At each scale i, freeze ε_{i-1} and solve the stationary Fokker-Planck equation for $f(\varepsilon_i|\varepsilon_{i-1})$.
- One finds that $f(\varepsilon_i|\varepsilon_{i-1})$ is the **G**eneralized **I**nverse **G**aussian:

$$f(\varepsilon_i|\varepsilon_{i-1}) = \frac{(\varepsilon_i/\varepsilon_{i-1})^{p-1}}{2\varepsilon_{i-1}\alpha^p K_p(\omega)}\exp\left(-\frac{\beta\varepsilon_i}{\varepsilon_{i-1}} - \frac{\beta\alpha^2\varepsilon_{i-1}}{\varepsilon_i}\right)$$

where $p = \beta(1-\alpha^2)$ and $\omega = 2\alpha\beta$.

Background Distribution

- The distribution $f(\varepsilon_N)$ at the smallest scale is

$$f_N(\varepsilon_N) = \int_0^\infty \dots \int_0^\infty f(\varepsilon_N|\varepsilon_{N-1}) \prod_{i=1}^{N-1} \left[f(\varepsilon_i|\varepsilon_{i-1}) d\varepsilon_i \right]$$

- The integral can be written in terms of a **new class of special functions**:

$$f(\varepsilon_N) = \frac{1}{\varepsilon_0 \left[\alpha K_p(\omega) \right]^N} R^{N,0}_{0,N} \left(\begin{array}{c} - \\ (\boldsymbol{p} - \boldsymbol{1}, \boldsymbol{\omega}/2) \end{array} \middle| \beta^N \frac{\varepsilon_N}{\varepsilon_0} \right),$$

where $R^{m,n}_{p,q}$ is the R-function (the 'Recife' function).

The R-Function

- The Meijer G-function is defined through its Mellin-Barnes transform:

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \,\middle|\, x\right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + s) \prod_{j=1}^{n} \Gamma(1 - a_j - s)}{\prod_{j=n+1}^{p} \Gamma(a_j + s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - s)} x^{-s} \, ds$$

- The R-function is a generalization of the G-function:

$$R_{p,q}^{m,n}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \,\middle|\, x\right) = \frac{1}{2\pi i} \int_\Gamma \frac{\prod\limits_{j=1}^{m} B_j^s K_{b_j+s}(2B_j) \prod\limits_{k=1}^{n} A_k^{-s} K_{1-a_k-s}(2A_k)}{\prod\limits_{k=n+1}^{p} A_k^s K_{a_k+s}(2A_k) \prod\limits_{j=m+1}^{q} B_j^{-s} K_{1-b_j-s}(2B_j)} x^{-s} \, ds$$

- In the limit $A_j \to 0$ and $B_j \to 0$, we have $R_{p,q}^{m,n} \to G_{p,q}^{m,n}$.

Velocity Distribution

- The marginal distribution of velocity increments is also obtained exactly:

$$P(\delta v_r) = \int_0^\infty \frac{1}{\sqrt{2\pi\varepsilon_\ell}} \exp\left[-\frac{(\delta v_r - \mu(\varepsilon_\ell - \varepsilon_0))^2}{2\varepsilon_\ell}\right] f(\varepsilon_\ell)d\varepsilon_\ell$$

$$\propto \int_0^\infty {\varepsilon_\ell}^{-1/2} \exp\left[-\frac{(\delta v_r - \mu(\varepsilon_\ell - \varepsilon_0))^2}{2\varepsilon_\ell}\right] R^{N,0}_{0,N}\left(\begin{matrix} - \\ (\boldsymbol{\alpha} - \mathbf{1}, \boldsymbol{\omega}/2)\end{matrix} \middle| \bar{\beta}\frac{\varepsilon_\ell}{\varepsilon_0}\right) d\varepsilon_\ell$$

$$\propto R^{N+1,0}_{0,N+1}\left(\begin{matrix} [(0, \boldsymbol{\alpha} - \frac{\mathbf{1}}{\mathbf{2}}), [(\frac{|\mu y|}{2}, \frac{\boldsymbol{\omega}}{2})] \\ - \end{matrix} \middle| \frac{\bar{\beta}y^2}{2\varepsilon_0}\right),$$

where

$$\boxed{y = \delta v_r + \mu\langle\varepsilon_r\rangle}$$

Asymptotic Behavior: Heavy Tails

- The asymptotic behaviour displays non-Gaussian tails:

$$\sim |y|^{\theta} \exp\left[-\beta N \left(\frac{y}{\varepsilon_0|\mu|}\right)^{1/N}\right] \times \begin{cases} 1 & \text{for } \delta v_r \to -\infty \\ e^{-2|\mu|y} & \text{for } \delta v_r \to +\infty \end{cases}$$

  with $\theta = p + 1/(2N) - 3/2$.

- The distribution has a modified **stretched exponential** in the left tail and an **exponential decay** in the right tail.

- **Generating the variance series:**

$$P(\delta v_r) = \int_0^\infty \frac{1}{\sqrt{2\pi\epsilon_r}} \exp\left[-\frac{(\delta v_r - \mu(\epsilon_r - \epsilon_0))^2}{2\epsilon_r}\right] f(\epsilon_r) d\epsilon_r$$

$$\delta v_\tau = v(t+\tau) - v(t)$$

Velocity increments series

$$\epsilon(t) = \frac{1}{M}\sum_{j=0}^{M-1}\left[\delta v(t - j\delta t) - \overline{\delta v}(t)\right]^2$$

Variance estimator in box of size *M*

$$\overline{\delta v}(t) = \frac{1}{M}\sum_{j=0}^{M-1}\delta v(t - j\delta t)$$

Velocity increments average in each box.

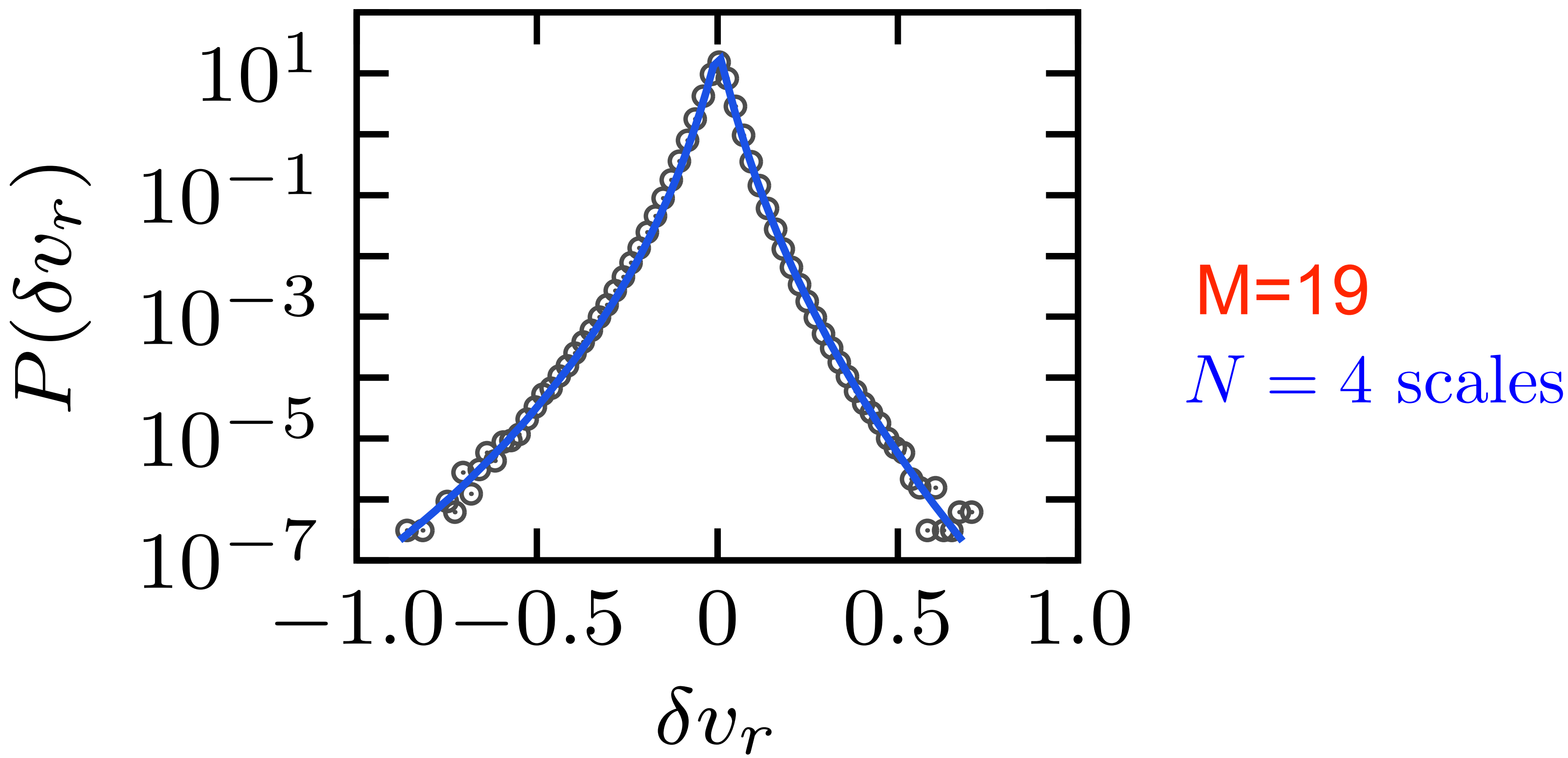
Increments series $\{ x_1,\ x_2,\ x_3,\ x_4,\ \ldots,\ x_{M-1},\ x_M,\ x_{M+1},\ x_{M+2},\ x_{M+3},\ \ldots,\ x_N \}$

So on

Variances series $\{\epsilon_1,\ \epsilon_2,\ \epsilon_3,\ \epsilon_5,\ \epsilon_6,\ \epsilon_7,\ \epsilon_8,\ \epsilon_9,\ \epsilon_{10},\ \ldots \}$

Optimal Window Size

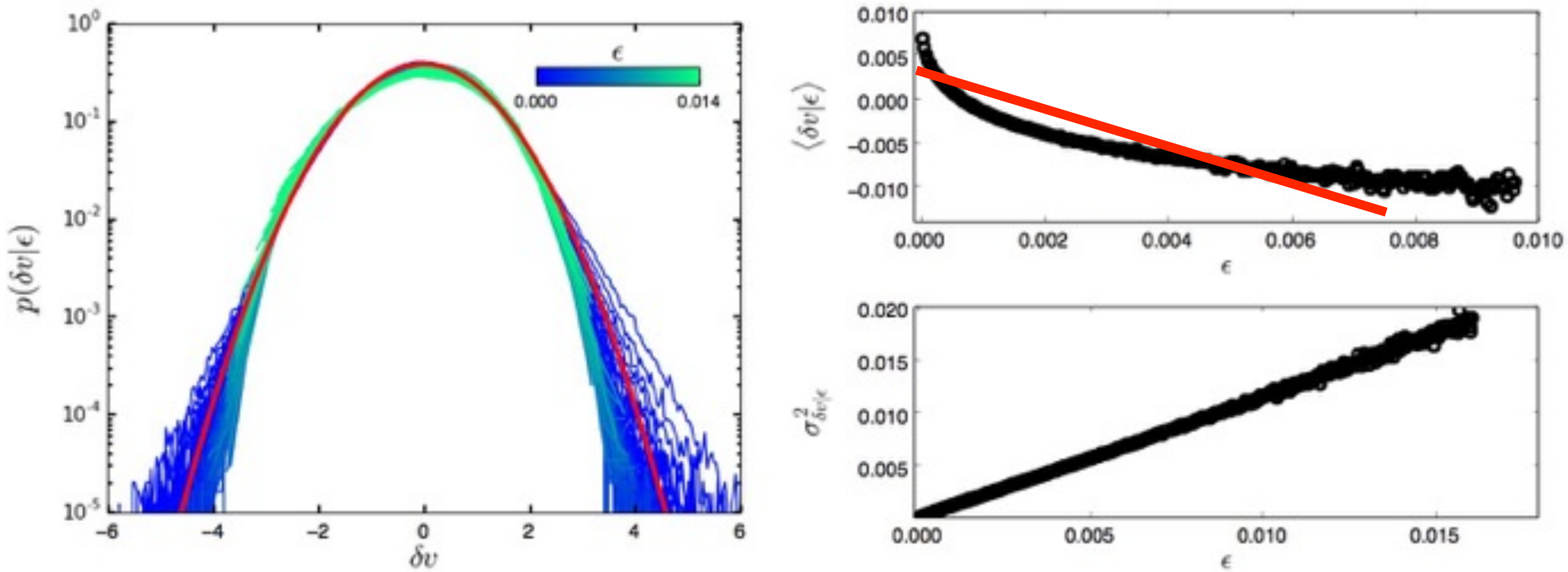
- Choose M and μ such that the compounding of the empirical distribution of $\{\epsilon(k)\}$ with the non-centered Gaussian best fits the distribution of velocity increments.



$$P(\delta v_r) = \int_0^\infty P(\delta v_r|\epsilon) f(\epsilon) d\epsilon \approx \frac{1}{N_M} \sum_{i=1}^{N_M} \frac{1}{\sqrt{2\pi\epsilon_i}} \exp\left\{ -\frac{\left[\delta v_r - \mu\left(\epsilon_i - \langle\epsilon\rangle\right)\right]^2}{2\epsilon_i} \right\}$$

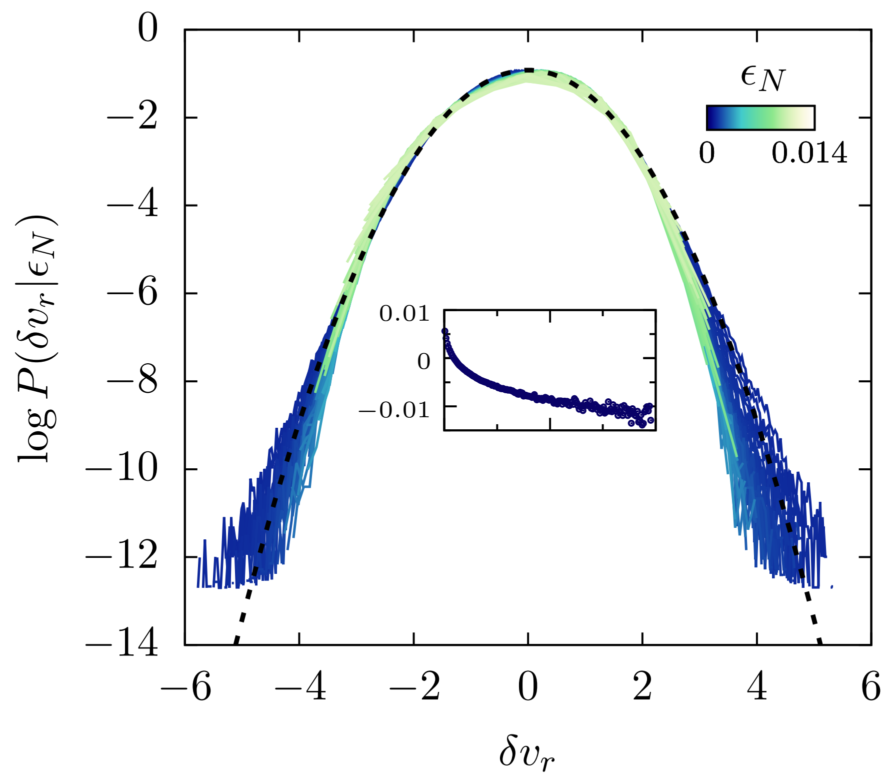
Comparison with DNS Data

- **The conditional distribution is well described by a Gaussian:**



- The conditional variance is indeed linear in the energy flux.
- The conditional mean, however, is only approximately linear.
- The important point is that the conditional mean decreases from positive to negative as the rate of energy transfer increases.

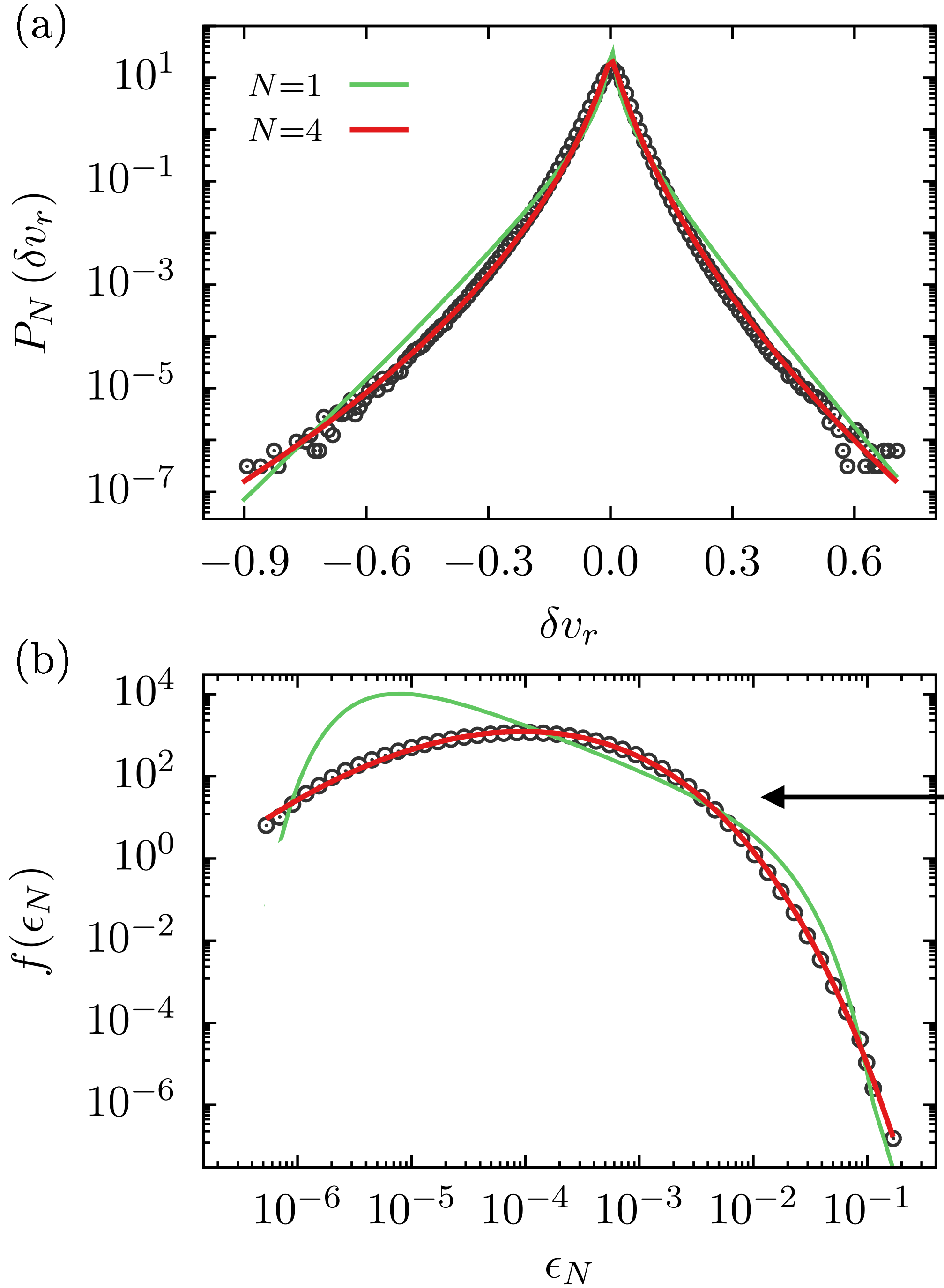
Origin of Skewness



- The emergence of skewness ultimately lies in the intermittency of the energy dissipation rate.
- In regions of small (large) ε the fluid particle is more likely to accelerate (decelerate) from one point to the next, resulting in a positive (negative) local average $\langle \delta v_r | \varepsilon \rangle$.
- For long times $\langle \delta v_r \rangle = 0$, but the skewness is negative.

Comparison with DNS Data

- **DNS with 1024^3 points:**



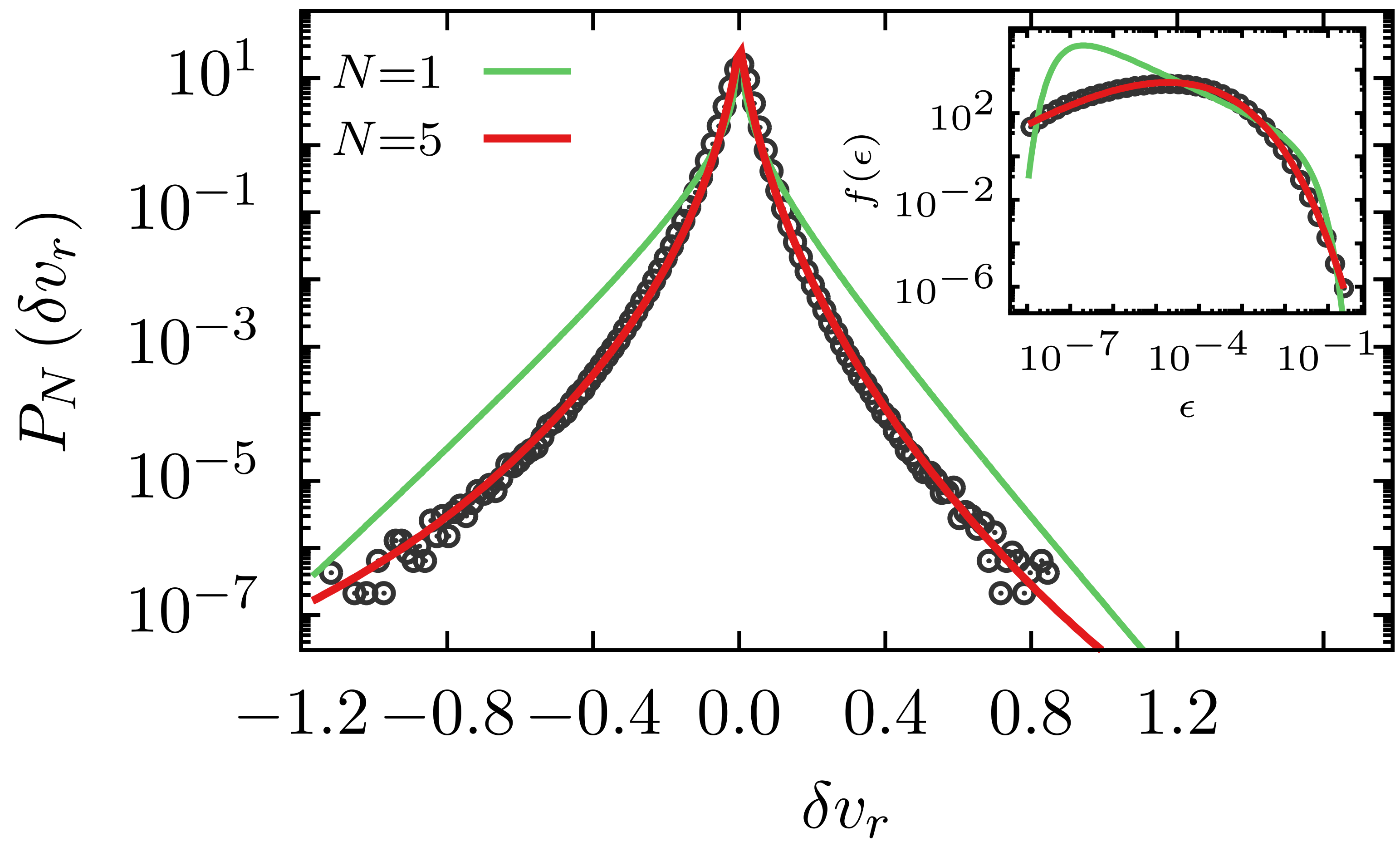
Here is just a plot with the parameters from below.

Fitting is done here!

$$\mathrm{Re}_\lambda \approx 418$$

Comparison with DNS Data

- **DNS with 4096^3 points:**



$N = 5$ scales

$\mathrm{Re}_\lambda \approx 610$

Conclusions

- **H-Theory**: unified description of fluctuating phenomena with asymmetric non-Gaussian tails.

- **Analytical expressions** for all probability distributions.

- **Wide range of application****s:**
  - Turbulence ✓
  - Financial markets ✓
  - Random Lasers ✓
  - Quantum systems
  - Anomalous diffusion
  - Relativistic gas
  - Other systems (?)

- **Future work:** investigation of the model for different scales r and Re.

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RAPID COMMUNICATIONS

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Multicanonical distribution: Statistical equilibrium of multiscale systems

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Universality classes of fluctuation dynamics in hierarchical complex systems

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Maximum entropy approach to H-theory: Statistical mechanics of hierarchical systems

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Obrigado.