Burger's Equation

J M Burgers 1974 The nonlinear diffusion equation - Reidel

E Hopf 1950 Commun. Pure Appl. Mathematics $\underline{3}$, 201.

$$\frac{Du}{Dt} \equiv \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \quad , \quad u(0,x) = u_0(x) \quad \text{(and boundary conditions)}$$

A. The deterministic problem $u_0(x) \to 0$ $|x| \to \infty$, u_0 smooth

B. The statistical problem

I Symmetries and Conservation laws

Invariance: space and time translations
space reflections $t \to t$, $x \to -x$, $u \to -u$
Galilean

Conservation: Momentum $P(t) = \int_{-\infty}^{\infty} u(t,x)\,dx$: $\frac{dP}{dt} = 0$

Energy $E_{TOT}(t) = \frac{1}{2}\int u^2\,dx$, $\Omega_{TOT} = \frac{1}{2}\int \left(\frac{\partial u}{\partial x}\right)^2 dx$

$$\frac{dE_{TOT}}{dt} = -2\nu\,\Omega_{TOT}$$

$$\frac{d}{dt}\int u^n\,dx = 0 \quad , \quad \forall n \quad \text{if } \nu = 0$$

$$\frac{d}{dt}\sup_x |u(t,x)| \le 0 \quad , \quad \frac{d}{dt}\int \left|\frac{\partial u}{\partial x}\right| dx \le 0 \quad , \quad \nu \ge 0$$

(These results do not hold for N-S equation because of the pressure term)

critical diffusivity such that $\alpha > \alpha_{cr}$ always get smooth solutions is $\alpha_{cr} = \frac{1}{2}$ (replace $\frac{\partial^2 u}{\partial x^2}$ by $\left(\frac{\partial^2 u}{\partial x^2}\right)^\alpha$)

Liberté • Égalité • Fraternité
RÉPUBLIQUE FRANÇAISE
CONSULAT GENERAL DE FRANCE A RIO DE JANEIRO

INSTITUT FRANÇAIS
BRASIL

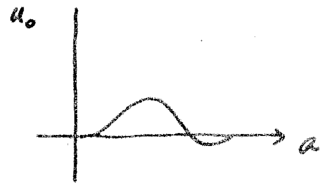
II : Consider $\nu = 0$.

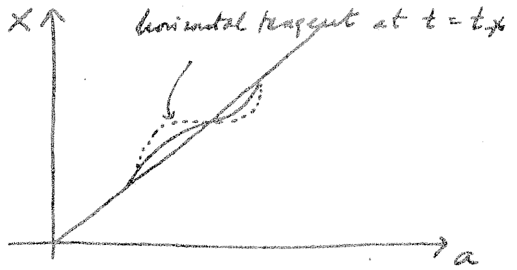
Then $\dfrac{Du}{Dt} = \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0$

Lagrangian coordinates $X(t,a)$ = position at time t of fluid particle originating from a
$= a + t u_0(a)$

Invertibility : $J = \dfrac{\partial X}{\partial a} = 1 + t u_0'(a)$, $u' = \partial u / \partial a$

So define $t_* = \left[- \inf u_0'(a) \right]^{-1}$





Suppose u_0' achieves minimum at a_0. w.l.o.g. of generality assume $a_0 = 0$. Then $u_0(a_0) \equiv u_0(0) = v$ can be transformed away by a Galilean transformation. So assume $v = 0$. Then

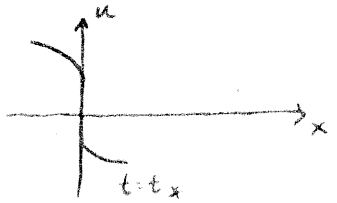
$$u_0(a) = a \underset{(<0)}{u_0'(0)} + \frac{a^3}{3!} \underset{(>0)}{u_0'''(0)} + O(a^4) \qquad \text{Assume that } u''' \neq 0$$

Then $X(t_*, a) = t_* \dfrac{a^3}{3!} u_0'''(0) + O(a^4)$

or $a = \left[\dfrac{6}{t_* u_0'''(0)} X \right]^{1/3} + O(X^{2/3})$

The velocity $u(t, X(t,a)) = u_0(a)$ is now given by

$$u(t_*, x) = u_0'(0) \left[\frac{6x}{t_* u_0'''(0)} \right]^{1/3} + O(x^{2/3})$$



NB. $\Omega_{TOT}(t) \propto (t_* - t)^{-1/2} \to \infty$ as $t \to t^{*-}$

$\hat{u}(t,k) \propto k^{-4/3}$ as $k \to \infty$

Fourier - Lagrangian representation

$$u(t,x) = \int e^{ikx}\, \hat{u}(t,k)\, dk, \qquad \hat{u}(t,k) = \frac{1}{2\pi}\int e^{-ikx}\, u(t,x)\, dx$$

$$\hat{u}(t,k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikX(t,a)}\, u_0(a)\, \frac{\partial X}{\partial a}\, da$$

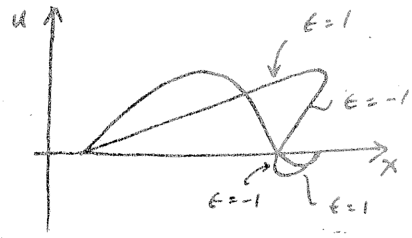
So for $k \neq 0$: $\quad = \dfrac{1}{2\pi i k}\displaystyle\int_{-\infty}^{\infty} e^{-ikX(t,a)}\, u_0'(a)\, da \quad , \quad X(t,a) = a + t u_0(a)$

NB. for Gaussian u_0 can evaluate all moments of $\hat{u}$. But always a fraction of the realizations will be beyond their t^* since arbitrarily large gradients (since they will also be Gaussian) have finite probability of occurrence.

Analytic continuation of $\hat{u}(t,k)$ in time for fixed k

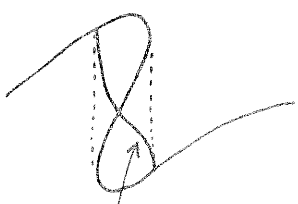
$$\hat{u}(t,k) = \frac{1}{2\pi i k}\sum_{0}^{\infty} \frac{(-ikt)^n}{n!}\int_{-\infty}^{\infty} e^{-ika}\, u_0^{(n)}(a)\, u_0'(a)\, da$$

has infinite radius of convergence, and so is entire function of t (NB. singularities are at $k=\infty$, and so are not seen in an expansion at fixed t)



The above function is the F.T. of the function $\sum_i \epsilon_i\, u_i(t,x)$

$\epsilon = 1$ on direct branch, $\epsilon = -1$ on retrograde branch. Sign comes from the Jacobian once we decompose the multivalued function. For this function $\hat{u}(t,k) \propto k^{-3/2}$. But what is this solution?



this is the "solution" obtained above, but is not a solution of Burgers' equation