Scaling and Exponents

We have discussed scaling

$$(*) \qquad \|\delta u(r;\cdot)\|_p \sim u_{rms}\left(\frac{|r|}{L}\right)^{\sigma_p}, \qquad |r| \le L$$

or equivalently, for $S_p(r) = \|\delta u(r)\|_p^p$,

$$(**) \qquad S_p(r) \sim u_{rms}^p \left(\frac{r}{L}\right)^{\zeta_p}, \qquad |r| \le L$$

We have not given a precise definition for σ_p and ζ_p. To this day, there is no proof of such scaling from first principles. However, when $u \in L^p$, we can define σ_p and ζ_p which always exist and agree with the scalings when those hold.

More precisely, a scaling $(*)$ implies the following limit exists

$$\sigma_p = \lim_{|r| \to 0} \frac{\ln \|\delta u(r;\cdot)\|_p}{\ln(|r|/L)}.$$

This means that a plot of $\ln \|\delta u(r;\cdot)\|_p$ vs. $\ln(|r|/L)$ should better approximate a straight line for $|r| \le L$ and getting smaller. This limit is not guaranteed to exist. Instead consider

$$\inf_{r<\ell} \frac{\ln \|\delta u(r)\|_p}{\ln(|r|/L)}$$

which is an increasing function of ℓ. Thus, the limit $\ell \to 0$ is guaranteed to exist (possibly $+\infty$). This is called the limit-infimum. Then

$$\underline{\sigma_p} = \liminf_{|r| \to 0} \frac{\ln \|\delta u(r)\|_p}{\ln(|r|/L)}$$

The lower exponent always exists. Means, $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that

$$\ell < \delta \implies \underline{\sigma}_p - \varepsilon \le \inf_{|r|<\ell} \frac{\ln \|\delta u(r)\|_p}{\ln(|r|/L)} \le \underline{\sigma}_p + \varepsilon$$

Equivalently, $\forall\, \ell < \delta$, $\exists\, r \le \ell$ such that

$$\left(\frac{r}{L}\right)^{\underline{\sigma}_p + \varepsilon} \le \|\delta u(r)\|_p \le \left(\frac{r}{L}\right)^{\underline{\sigma}_p - \varepsilon}$$

In the same manner, we define

$$\underline{\zeta}_p = \liminf_{|r|\to 0} \frac{\ln S_p(r)}{\ln(r/L)}$$

so that $\underline{\zeta}_p = p\,\underline{\sigma}_p$. For high-Re turbulence, this definition is suitable if one considers limit $\nu \to 0$ first (if it exists!) In this case, an infinitely long inertial range exists for singular velocity fields. If one keeps $\nu > 0$ finite, then $\sigma_p = 1$ and $\zeta_p = p$ for all p. (provided the NS solution remains smooth).

At finite Re, one can hope to get $\sigma_p < 1$ or $\zeta_p < p$ only in an intermediate asymptotic sense, i.e. there should exist a scale

$$\eta_p = (\text{const})\, L\, Re^{-\alpha_p}, \qquad \alpha_p > 0$$

such that

$$\inf_{r<\ell} \frac{\ln \|\delta u(r)\|_p}{\ln(r/L)} \to \sigma_p$$

should hold better and better for decreasing ℓ in range $\eta_p \le \ell \le L$. The key is that this intermediate range should be longer and longer as Re

Here are some exact results:

<u>Proposition:</u> ζ_p is a concave function of $p \in [0,\infty)$.

<u>Proof:</u> We show that, $\forall t \in [0,1]$,

$$\zeta_{tp+(1-t)p'} \ge t\,\zeta_p + (1-t)\,\zeta_{p'} \qquad \text{for } p, p' \ge 0.$$

$$\begin{aligned}
S_{tp+(1-t)p'}(r) &= \langle |\delta u(r)|^{tp+(1-t)p'} \rangle \\
&= \langle |\delta u(r)|^{tp}\, |\delta u(r)|^{(1-t)p'} \rangle \\
&\le \langle |\delta u(r)|^{tp\cdot\frac{1}{t}} \rangle^{t} \cdot \langle |\delta u(r)|^{(1-t)p'\frac{1}{1-t}} \rangle^{1-t} \\
&= \langle |\delta u(r)|^{p} \rangle^{t} \langle |\delta u(r)|^{p'} \rangle^{1-t} = [S_p(r)]^t [S_{p'}(r)]^{1-t}.
\end{aligned}$$

Thus, for $r<L$,

$$\frac{\ln S_{tp+(1-t)p'}(r)}{\ln(r/L)} \ge t\,\frac{\ln S_p(r)}{\ln(r/L)} + (1-t)\,\frac{\ln S_{p'}(r)}{\ln(r/L)}. \quad \text{Take limit } r\to 0$$

Corollary: The exponent $\sigma_p = \zeta_p/p$ is non-increasing in p.

Proof: Note first that $\zeta_0 = 0$ since $S_0(r) = \langle |\delta u(r)|^0 \rangle = \langle 1 \rangle = 1$. Thus, take any $p' \geq p$ and write

$$p = \left(1 - \frac{p}{p'}\right) \cdot 0 + \frac{p}{p'} p'$$

$$\Rightarrow \quad \zeta_p \geq \left(1 - \frac{p}{p'}\right) \zeta_0 + \frac{p}{p'} \zeta_{p'} = \frac{p}{p'} \zeta_{p'}$$

$$\Rightarrow \quad \sigma_p = \frac{\zeta_p}{p} \geq \frac{\zeta_{p'}}{p'} = \sigma_{p'}.$$

Proposition: If u is bounded, then ζ_p is non-decreasing in p.

Proof: If $u \in L^\infty$, then $|\delta u(r;x)| \leq 2\|u\|_{L^\infty}$. Then, taking $p' \geq p$,

$$\langle |\delta u(r)|^{p'} \rangle = \langle |\delta u(r)|^{p'-p} \cdot |\delta u(r)|^p \rangle$$
$$\leq (2\|u\|_{L^\infty})^{p'-p} \langle |\delta u(r)|^p \rangle$$

Thus,

$$\frac{\ln \langle |\delta u(r)|^{p'} \rangle}{\ln (r/L)} \geq \frac{\ln \langle |\delta u(r)|^p \rangle}{\ln (r/L)} + (p'-p) \frac{\ln (2\|u\|_{L^\infty})}{\ln (r/L)}$$

The last term vanishes as $r \to 0$, thus taking limit as $r \to 0$ we find $\zeta_{p'} \geq \zeta_p$.

Corollary: If u is bounded, then $\sigma_p \geq 0$ for all $p > 0$.

Proof: For $p > 0$, $\zeta_p \geq \zeta_0 = 0$. Thus $\sigma_p = \frac{\zeta_p}{p} \geq 0$.

Exercise: Show if $u \in L^p$, then $\sigma_p \geq 0$.

Remark: It follows from these two corollaries that

$$\lim_{p \to \infty} \sigma_p = \sigma_\infty \geq 0$$

must exist, provided u is bounded. Usually this is denoted

$$\sigma_\infty = h_{min}$$

because it represents the minimal Hölder singularity in the flow.

Note that the most singular behavior in the flow determines h_{min}.
For Burgers, $\sigma_p = 1/p$ for $p \geq 1$, so $\sigma_\infty = \lim_{p\to\infty} \sigma_p = 0$, corresponding to shock discontinuities in the velocity.
The relation $\sigma_p \to h_{min}$ can be stated as (provided $h_{min} > 0$).

$$\zeta_p \sim p\, h_{min} \quad \text{as} \quad p \to \infty$$

Finally, we make some remarks about K41 theory's special status among scaling laws. Consider

$$S_p(r) \sim C_p\, u_{rms}^p \left(\frac{r}{L}\right)^{\zeta_p}$$

Using $u_{rms} \approx (\langle\varepsilon\rangle L)^{1/3}$ and setting $\zeta_p = \frac{p}{3} + \delta\zeta_p$ gives

$$S_p(r) \sim C_p\, (\langle\varepsilon\rangle r)^{p/3} \left(\frac{r}{L}\right)^{\delta\zeta_p}$$

Thus, K41 ($\zeta_p = p/3$) is unique in that it is the only possible scaling in which $S_p(r)$ is independent of L, depending only on $\langle\varepsilon\rangle$.

More generally, $S_p(r)$ should be expected to depend on both $\langle\varepsilon\rangle$ and L, even for $r \ll L$. This means that the small-scales remember, not only the flux from large scales, but also $\ln(L/r) = N$ the number of "cascade steps" going from L to r.

Multifractal Model

Formalism that relates ζ_p to other measurable quantities.

Local-Hölder exponents ~~[illegible]~~

Recall that u is h-Hölder at a point x if

$$|u(x+r) - u(x)| \leq |r|^h \quad \forall \ r.$$

Now we ~~[illegible]~~ define a local exponent $h(x)$. This should say

$$|\delta u(r;x)| \sim u_{rms} \left(\frac{\ell}{L}\right)^{h(x)} \qquad \text{when} \quad \ell \ll L,$$

or more precisely

$$\lim_{|r| \to 0} \frac{\ln |\delta u(r;x)|}{\ln (r/L)} = h \equiv h(x).$$

However, this limit need not exist! We follow a similar strategy...

$$\underline{h}(x) = \liminf_{|r| \to 0} \frac{\ln |\delta u(r,x)|}{\ln(r/L)}$$

is the maximal Hölder regularity exponent of u at x, i.e. u is Hölder with $h = h(x) - \varepsilon$ at x $\forall \varepsilon > 0$, but not with $h(x) + \varepsilon$.

Note that the limit supremum

$$\bar{h}(x) = \limsup_{|r| \to 0} \frac{\ln |\delta u(r;x)|}{\ln (r/L)}$$

defines a local singularity exponent, i.e. $\forall \varepsilon > 0$, $\exists \ \delta > 0$ s.t. when $\ell < \delta$

$$|\delta u(r;x)| > (r/L)^{\bar{h}(x) + \varepsilon} \quad \text{for all} \quad r < \ell$$

$$|\delta u(r;x)| < (r/L)^{\bar{h}(x) - \varepsilon} \quad \text{for some} \quad r < \ell.$$

These definitions are adequate for $h \in [0,1)$, but not otherwise.
If $h > 1$, then second or higher order differences can be used.
If $h \leq 0$, then wavelets can be used. See

H. Triebel Theory of Function Spaces, Section 2.5.10 - 12

Fractal Dimensions

We now consider notions of non-integer dimensionality for sets $S \subset \mathbb{R}^d$.

First consider <u>box-counting dimension</u> $D_B(S)$. If $S \subset \mathbb{R}^d$ is bounded,

$N_k(S)$ = number of hypercubes in a regular grid of sidelength 2^{-k} which intersect S.

Define

$$\overline{D}_B(S) = \limsup_{k\to\infty} \frac{1}{k} \log_2 (N_k(S)) = \text{upper fractal dimension}$$

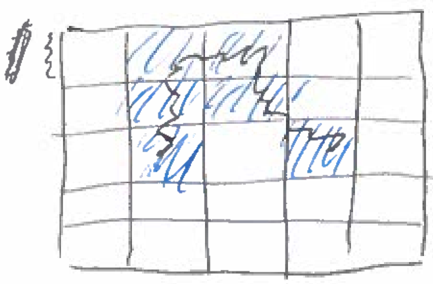
$$\underline{D}_B(S) = \liminf_{k\to\infty} \frac{1}{k} \log_2 (N_k(S)) = \text{lower fractal dimension}$$

so that $\underline{D}_B(S) \le \overline{D}_B(S)$. If $\overline{D}_B(S) = \underline{D}_B(S) = D_B$, we say S has fractal dimension D_B and $N_k(S) \sim 2^{k D_B(S)}$. Notice that the total number of boxes per unit volume grows as $N_k \sim 2^{kd}$. Hence the fraction of boxes intersecting S goes as

$$\frac{N_k(S)}{N_k} \sim 2^{-k(d - D_B(S))} \sim 2^{-k\, K_B(S)}$$

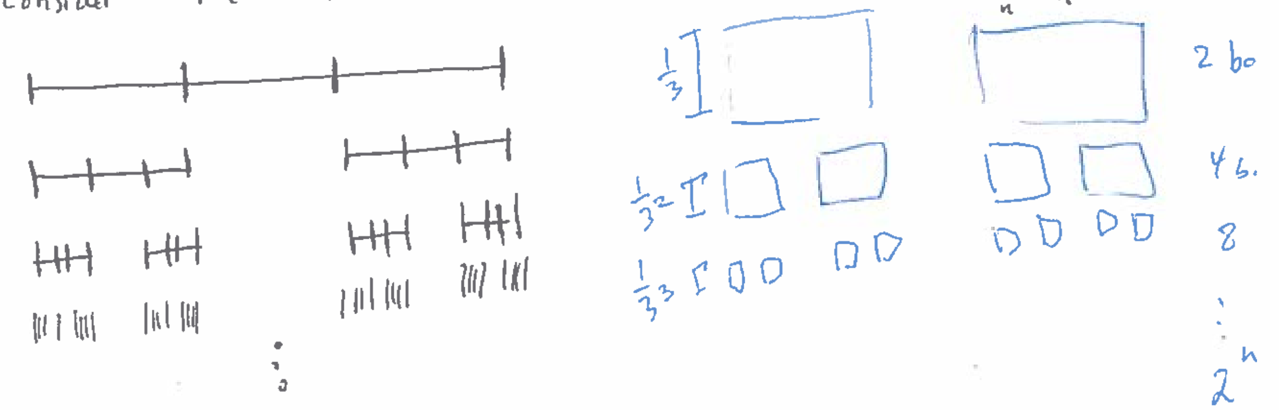
where

$$0 \le K_B(S) = d - D_B(S) = \text{fractal codimension of } S.$$



Remark: Instead of boxes with sidelength 2^{-k}, decreasing by factors of 2, one can use any factor $\lambda > 0$ with sidelengths λ^{-k}. In this case $N_k(S) \sim \lambda^{k D_B(S)}$ with same value of $D_B(S)$ independent of λ

<u>Example:</u> Consider the middle-thirds Cantor set $K = \bigcap_n K_n$:



Choosing $\lambda = 3$, we get (by direct counting)

$$N_k(K) = 2^k$$

so

$$D_B(K) = \lim_{k\to\infty} \frac{\log_3(N_k(K))}{k}$$

$$= \lim_{k\to\infty} \frac{\log_3(2^k)}{k} = \log_3(2) = \frac{\ln(2)}{\ln(3)} \simeq 0.630$$

There are many other notions of fractal dimension, e.g. <u>Hausdorff dimension</u> $D_H(S)$. The difference in definition essentially involves covering sets with balls of size $\leq 2^{-k}$ not just $= 2^k$, so that the number of balls required is always fewer (maybe much fewer, for example a countable set has Hausdorff measure 0 but such sets may be everywhere dense, having box counting dimension d). Thus

$$D_H(S) \leq \underline{D}_B(S) \leq \overline{D}_B(S).$$

See

K. Falconer, Fractal Geometry, 1990.