Navier - Stokes Turbulence

(also see Eyink's course Notes, found on his website
Uriel Frisch, Turbulence

Incompressible NS equation:

$$\partial_t u + u\cdot\nabla u = -\nabla p + \nu\Delta u$$
$$\nabla\cdot u = 0$$
$$u|_{\partial\Omega} = 0 \qquad \text{(no slip)}$$

Pressure is non-local, solving

$$-\Delta p = (\nabla u)^T : \nabla u$$
$$\hat{n}\cdot\nabla p = \nu\,\hat{n}\,\Delta u$$

Define the Reynolds number

$$Re = \frac{UL}{\nu}, \qquad L = \text{domain}, \qquad U = \text{characteristic velocity}$$

Non-dimensionalization, $\bar{u} = u/U$, $\bar{x} = x/L$, $\bar{t} = (U/L)\,t$,

$$\partial_{\bar{t}}\bar{u} + \bar{u}\cdot\bar{\nabla}\bar{u} = -\bar{\nabla}\bar{p} + \frac{1}{Re}\bar{\Delta}\bar{u}.$$

Principle of hydrodynamic similarity: Two flows with same geometry but different scale are essentially identical if Re are same.

Infinite-Re limit → "ideal turbulent state" ⟺ zero-viscosity limit

Energy in NS fluids: smooth solutions satisfy

$$\frac{d}{dt}\int_\Omega |u|^2\,dx = -\nu\int_\Omega |\nabla u|^2\,dx$$

Remarkable observation

$$\nu\int|\nabla u|^2\,dx \longrightarrow \varepsilon(t) > 0 \quad \text{as } \nu\to 0.$$

Pictures!

How is this possible? Clearly $|\nabla u| \to \infty$ as $\nu \to 0$, but Onsager said something more refined:

> For energy dissipation to take place without the final assistance of viscosity, the velocity cannot possess > "1/3rd" of a derivative. He stated in terms of Hölder continuity
>
> $$|u(x+r) - u(x)| \leq |r|^{\alpha}$$
>
> cannot hold with $\alpha > 1/3$.

Let's understand how Onsager arrived at this remarkable conclusion. Introduce coarse-graining / filtering. Smooth kernel G

"Bump function"

$$G(x) = \begin{cases} N \exp\left(-\frac{1}{1-|x|^2}\right) & |x|<1 \\ 0 & |x| \geq 1 \end{cases} \qquad G \in C_0^{\infty}$$

$$G(x) \geq 0$$
$$G(x) \to 0 \quad \text{rapidly as } |x| \to \infty$$
$$\int G(x)\,dx = 1.$$

Set $G_\ell(x) = \ell^{-d} G(x/\ell)$ for $\ell > 0$. Coarse-grained velocity

$$\bar{u}_\ell = \int dr \; G_\ell(r)\, u(x+r).$$

This represents the average velocity of a fluid parcel of size ℓ at position x. Note that if $u \in L^p$, then $\|\bar{u}_\ell - u\|_{L^p} \xrightarrow{\ell \to 0} 0$.

~~[illegible]~~ The coarse-grained field does not satisfy NS equations, but rather

$$\partial_t \bar{u}_\ell + \bar{u}_\ell \cdot \nabla \bar{u}_\ell = -\nabla \bar{p}_\ell + \nu \Delta \bar{u}_\ell - \nabla \cdot \tau_\ell(u,u)$$
$$\tau_\ell(u,u) = \overline{(u \otimes u)}_\ell - \bar{u}_\ell \otimes \bar{u}_\ell$$
$$\nabla \cdot \bar{u}_\ell = 0$$

force representing action of small unresolved scales of flow.

<u>Remark</u>; fixing ℓ

$$\nu \|\Delta \bar{u}^{\nu}_\ell\|_{L^p} \leq \frac{\nu}{\ell^2} \left\| \int dr\, (\Delta G)_\ell(r)\; u^{\nu}(\cdot + r) \right\|_{L^p} \leq \frac{\nu}{\ell^2} \|\Delta G\|_{L^1} \|u^{\nu}\|_{L^p} \xrightarrow{\nu \to 0} 0 \quad \text{if } \|u^{\nu}\|_{L^p} < C.$$

"Navier-Stokes $\to$ weak Euler" (requires also $u^{\nu} \xrightarrow{L^2_{t,x}} u$).

Dynamics of resolved energy:

$$E_\ell^\nu(t) = \frac{1}{2}\int |\bar{u}_\ell^\nu(x,t)|^2 dx .$$

Note

$$\partial_t\left(\frac{1}{2}|\bar{u}_\ell|^2\right) + \nabla\cdot\left(\left(\frac{1}{2}|\bar{u}_\ell|^2 + \bar{p}_\ell\right)\bar{u}_\ell + \tau_\ell\cdot\bar{u}_\ell - \nu\nabla\left(\frac{1}{2}|\bar{u}_\ell|^2\right)\right)$$
$$= -\nu|\nabla\bar{u}_\ell|^2 + \nabla\bar{u}_\ell : \tau_\ell(u,u) + \bar{u}_\ell\cdot\bar{f}_\ell .$$

$$\Rightarrow \quad \frac{d}{dt}E_\ell^\nu(t) = -\nu\int|\nabla\bar{u}_\ell|^2 dx - \int \Pi_\ell \, dx + \int \bar{u}_\ell\cdot f \, dx$$

where $\Pi_\ell[u] = -\nabla\bar{u}_\ell : \tau_\ell(u,u)$.

Note, as $\nu\to 0$

$$\left|\nu\int|\nabla\bar{u}_\ell|^2 dx\right| \le \frac{\nu}{\ell^2}\|u^\nu\|_{L^2}^2 \le \frac{\nu}{\ell^2} \to 0.$$

~~Provided that u^ν ... L^2 ..., then~~

~~Now, assume that $\{u^\nu\}_{\nu>0}$ is uniformly bounded in $L^3(0,T;B_3^{s,\infty})$ for $s>1/3$.~~

recall

$\mathcal{B}$

Note that

$$\nabla\bar{u}_\ell = -\frac{1}{\ell}\int(\nabla G)_\ell(r)\, u(x+r)\,dr$$
$$= -\frac{1}{\ell}\int(\nabla G)_\ell(r)\,\delta u(r;x)\,dr$$

and

$$\tau_\ell(u,u) = \overline{(u\otimes u)}_\ell - \bar{u}_\ell\otimes\bar{u}_\ell$$
$$= \overline{((u-v)\otimes(u-v))}_\ell - (\bar{u}_\ell - v)\otimes(\bar{u}_\ell - v), \quad \text{e.g. } v = u(r$$
$$= \langle\delta u(r;x)\otimes\delta u(\cdot;x)\rangle_G - \langle\delta u(\cdot;x)\rangle_G\otimes\langle\delta u(\cdot;$$

Constantin - E - Titi (1994) commutator estimate.

Now suppose that

$$(*) \quad \int_0^T \|u^\nu\|_{L^3}^3 \, dt < C_0 \qquad \text{and} \qquad S_3^u(r) = \int |\delta u^\nu(r;x)|^3 dx := \|\delta u^\nu(r;\cdot)\|_{L^3}^3$$

satisfies

$$(S_3^u(r))^{1/3} = \|\delta u^\nu(r;\cdot)\|_{L^3} \le C_1(t)\, |r|^s$$

where $\int_0^T C_1^3(t)\, dt < \infty$.

This is equivalent to

$$u^\nu \in L^3(0,T; B_3^{s,\infty}(\mathbb{T}^d)), \quad \text{uniformly bounded.}$$

This is an L^p generalization of Hölder continuity called Besov regularity

If the NS solution obeys $(*)$, then

Hölder inequality $\|f_1 f_2 f_3\|_{L^1} \le \|f_1\|_{L^3}\|f_2\|_{L^3}\|f_3\|_{L^3}$, $1 = \frac{1}{3}+\frac{1}{3}+\frac{1}{3}$

$$\left| \iint_0^T \Pi_\ell[u]\, dx\, dt \right| \le \int_0^T \|\Pi_\ell[u]\|_{L^1} \, dt \lesssim C \frac{1}{\ell} \int_0^T \sup_{|r|<\ell} \|\delta u(r;\cdot)\|_{L^3}^3 \, dt$$

$$\le C \ell^{3s-1} \int_0^T C_1^3(t)\, dt \lesssim \ell^{3s-1}.$$

Thus, provided that $s > 1/3$, the energy flux through scale vanishes asymptotically as $\ell \to 0$. There can be no direct cascade sustained to infinitesimally small scales $\Rightarrow$ no anomalous dissipation!

$$\frac{1}{2}\int |u(t)|^2 dx = \frac{1}{2}\int |u_0|^2 dx \qquad \forall \quad t \in [0,T).$$

Physical conclusion: If the observations of anomalous dissipations are correct, and persist at higher Re, then the velocity u^ν cannot be uniformly bounded in $L^3(0,T; B_3^{s,\infty})$, i.e. singularities with $\le 1/3$ of a derivative in L^3 must emerge!

The conclusions of the argument can be strengthed.

Define pth order structure functions

$$S_p(r) = \|\delta u(r;x)\|_p^p$$

Then, if

$$\int_0^T \|u\|_{L^p}^3 \, dt < C \quad \text{and} \quad S_p(r)^{1/p} \le C|r|^{\sigma_p} \quad \text{with} \quad C \in L^3[0,T],$$

then we say $u \in L^3(0,T; B_p^{\sigma_p,\infty}(\mathbb{T}^d))$.

Onsagers argument say anomalous dissipation requires that

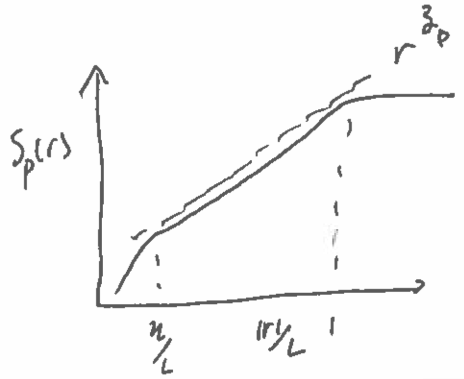
$$\boxed{\sigma_p \le 1/3 \quad \text{for all} \quad p \ge 3.}$$

Exersize: convince yourselfes of this (generalized Hölder inequality).

Notation:

$$\zeta_p = p\sigma_p, \qquad S_p(r) \le C|r|^{\zeta_p} \sim A_p u_{rms}^p \left(\frac{|r|}{L}\right)^{\zeta_p}$$

↑ often observed as a scaling



Thus:

$$\boxed{\langle \varepsilon \rangle \to 0 \quad \text{unless} \quad \zeta_p \le \frac{p}{3} \quad \text{for} \quad p \ge 3}$$

The classical Kolmogorov 1941 theory assumes

$$\zeta_p = \frac{p}{3} \quad \text{for all } p.$$

"Minimal singularity" sufficient to dissipate energy.

Then using

$$\varepsilon \sim \frac{u_{rms}^3}{L}$$

one gets

$$S_p(r) \sim C_p \, (\varepsilon |r|)^{p/3} \qquad \forall \; p \text{ and } |r| \ll L.$$

The physical meaning of assuming $\zeta_p = p/3$ is a "uniformity" assumption on the velocity increments $|\delta u(r;x)|$ which rules out large fluctuations in values for different x.

Note that the estimate $\delta u(\ell) \sim (\varepsilon \ell)^{1/3}$ is consistent with

$$\langle \Pi_\ell \rangle \sim \frac{\delta u^3(\ell)}{\ell} \sim \varepsilon \qquad \forall \; \ell \ll L.$$

Thus one can explain the observed rate of energy dissipation ε independent of viscosity ν, by the efficient transfer of energy down to small-scales where viscosity is effective.

This length-scale is the so-called Kolmogorov (micro) scale η. It can be obtained as the length-scale at which

$$\Pi_\ell \sim \nu \, |\nabla \bar{u}_\ell|^2 = \mathcal{O}\left(\nu \frac{\delta u^2(\ell)}{\ell^2}\right)$$

Using $\Pi_\ell = \mathcal{O}\left(\frac{\delta u^3(\ell)}{\ell}\right)$, we get an estimate for η as the soln of

$$\frac{\delta u^3(\eta)}{\eta} \cong \nu \frac{\delta u^2(\eta)}{\eta^2} \implies \delta u(\eta)\, \eta \cong \nu.$$

Note this implies $Re_\eta = \frac{\delta u(\eta)\eta}{\nu} \approx 1$. Using K41 conjecture $\delta u(\eta) \sim (\varepsilon\eta$

$$\varepsilon^{1/3} \eta^{4/3} \cong \nu \implies \eta \cong \nu^{3/4} \varepsilon^{-1/4}.$$

Often discussed in Fourier space

$$\text{K41} : \quad E_u(k) \sim \varepsilon^{2/3} k^{-5/3}.$$

$s = \frac{1}{2}(n$

<u>Remark</u>: (Wiener-Khinchin) $n \in (1,3)$, $E_f(k) \le C|k|^{-n} \iff S_2^f(r) \le C|r|^{n-1} \iff u \in B_2^{s,}$

Consequence of Anomalous dissipation & requisite irregularity: smooth sollutions of the Euler equations cannot govern a turbulent state. Onsager conjectured that weak solutions could: testing eqn with $\phi \in C_0^\infty([0,T) \times \mathbb{T}^d)$,

$$-\int_0^T\int u \cdot \partial_t \phi \, dx\, dt - \int u_0 \cdot \phi_0 \, dx$$
$$= -\int_0^T\int \nabla\phi : u \otimes u \, dx\, dt - \int_0^T\int \nabla\cdot\phi \, p \, dx\, dt.$$

Rm: This is equivalent to a "coarse-grained solution"

$$\partial_t \bar{u}_\ell + \nabla \cdot \overline{(u\otimes u)}_\ell = -\nabla \bar{p}_\ell$$

for any smooth mollifier G.

Onsagers conjecture states (for Hölder $C^h = B_\infty^{h,\infty}$).

(i) (weak) $u \in C^h$, $h > 1/3 \Rightarrow$ energy is conserved (proved!)
(Eyink 94, CET, 9

(ii) (strong) $\exists u \in C^h$, $h \leq \frac{1}{3}$ s.t. energy is decreasing
(De Lellis - Szekelyhidi, .. Isett 2016)

(iii) (strongest) Such solutions (in (ii)) should appear as $\nu \to 0$. Open!

Extensions: Duchon & Robert (2000) proved, under the hypothesis that $\{u^\nu\}_{\nu>0}$ cmp in $L^3_{x,t}$,

$$D_\ell[u] = \frac{1}{4\ell}\int (\nabla G)_\ell(r) \cdot \delta u(r) |\delta u(r)|^2 dr$$

has a distributional limit.

$$\lim_{\ell \to 0} D_\ell[u] = -D[u] = -\varepsilon[u] = \lim_{\nu \to 0} \nu \int |\nabla u^\nu| \, dx$$

Local form of cascade!

Weak solutions provide the mathematical framework for studying ideal turbulence in the inviscid limit:

1) local 4/5th law (Eyink 2002)

$$\lim_{|\ell|\to 0} \frac{1}{|\ell|} \int_{S^{d-1}} (\delta u(r;x)\cdot \hat{r})^3 \, d\omega(\hat{r}) = -\frac{12}{d(d+2)} \, \varepsilon(x,t)$$

2) locality (in scale) of cascade

(Eyink 2005, Cheskidov, Constantin, Friedlander, Shvydkoy 2008)

3) Intermittency and anomalous scaling (Parisi-Frisch, Mandelbrot)

TOY MODEL: Burgers equation

We now consider a simple 1-dimensional PDE model that has a non-vanishing energy dissipation as $\nu \to 0$, but for which the "minimal singularity" K41 prediction fails.

Model; proposed by Burgers in 1948, reads

$$\partial_t u + u\,\partial_x u = \nu\, \partial_x^2 u \qquad x \in \mathbb{R} \text{ or } \mathbb{T}$$

Note, can also be written as

$$\partial_t u + \frac{1}{2}\partial_x (u^2) = \nu\, \partial_x^2 u$$

so that momentum $\int u\,dx$ is conserved. The energy

$$\partial_t\left(\tfrac{1}{2}u^2\right) + \partial_x\left(\tfrac{1}{3}(u)^3 - \nu\, \partial_x\left(\tfrac{1}{2}u^2\right)\right) = -\nu(\partial_x u)^2$$

Thus

$$\frac{d}{dt}E(t) := \frac{d}{dt}\int \frac{1}{2}u^2(t)\,dx = -\nu \int (\partial_x u)^2\,dx .$$

Formally, as $\nu \to 0$

$$\nu \int (\partial_x u)^2 dx \to 0 \quad \text{and} \quad E(t) = E(0), \text{ energy conserved.}$$

However, this is not what occurs!

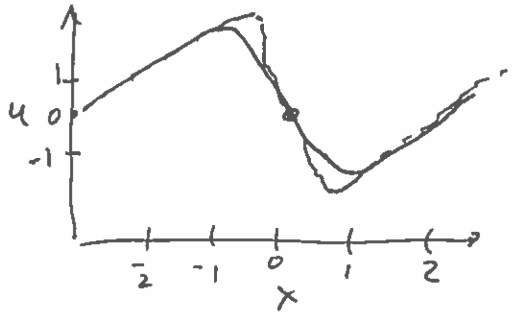
Consider a simple exact solution of 1D Burgers:

$$u^\nu(x,t) = \frac{1}{t}\left[x - L\tanh\left(\frac{Lx}{2\nu t}\right)\right].$$

called "Khokhlov saw-tooth solution", so called since

$$u^\nu(x,t) \sim \begin{cases} \frac{1}{t}(x+L) & \text{as } x \to -\infty \\ \frac{1}{t}(x-L) & \text{as } x \to +\infty \end{cases}$$

with discontinuity $\Delta u = \frac{2L}{t}$ across the origin.



at time $t=1$

In the limit as $\nu \to 0$

$$u(x,t) = \begin{cases} \frac{x+L}{t} & -L \leq x < 0 \\ \frac{x-L}{t} & 0 < x \leq L \end{cases}$$

a function on $[-L, L]$ with $u(\pm L, t) = 0$ and a jump discontinuity of size $\Delta u = \frac{2L}{t}$ at $x = 0$. This is called a <u>shock</u>.

Now we check for anomalous dissipation. First from shock solution directly

$$\int u^2(t)\,dx = \frac{1}{2L}\int_{-L}^{L}\frac{1}{2}u^2(t)\,dx = \frac{1}{2L}\int_0^L\left(\frac{x-L}{t}\right)^2 dx = \frac{1}{6}\left(\frac{L}{t}\right)$$

Thus $$\langle\varepsilon(t)\rangle = -\frac{d}{dt}\int\frac{1}{2}u^2(t)\,dx = \frac{1}{3}\frac{L^2}{t^3} = \frac{(\Delta u)^2}{12t} > 0.$$

<u>Exersize</u>: Check the result by computing $\nu \to 0$ limit of dissipation directly from $\varepsilon^\nu = \nu|\partial_x u|^2 \approx \frac{L^4}{4\nu t^4}\operatorname{sech}^4\left(\frac{Lx}{2\nu t}\right)$.

Burgers equation displays the phenomenon of anomalous dissipation, just like experiments and simulations of NS turbulence!

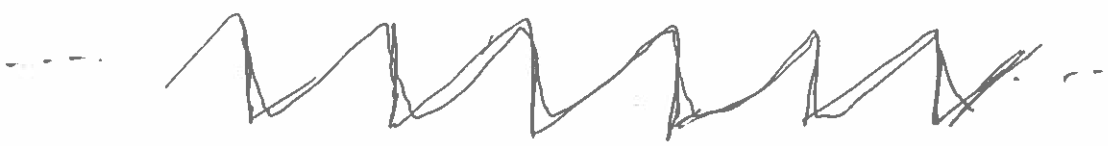
Moreover, all our previous analysis applies, and by Onsagers arguement, we deduce that

$$\zeta_p \le \frac{p}{3} \quad \text{for all } p \ge 3.$$

However, the "minimal singularity" theory of Kolmogorov doesn't apply here! To see this, consider the limiting shock profile

$$u(x,t) = \begin{cases} \frac{x+L}{t} & -L \le x < 0 \\ \frac{x-L}{t} & 0 < x \le L \end{cases}$$

which can be periodically extended to $\mathbb{R}$ with period $2L$



We can see that

$$u(x+\ell) - u(x,t) = \begin{cases} \ell/t & \text{if } 0 \notin [x, x+\ell] \\ (2L+\ell)/t & \text{if } 0 \in [x, x+\ell] \end{cases}$$

$\Rightarrow$

$$\langle |\delta u(\ell)|^p \rangle = \frac{1}{2L} \int_{-L}^{L} |u(x+\ell) - u(x)|^p \, dx$$

$$= \left(1 - \frac{\ell}{2L}\right)\left(\frac{\ell}{t}\right)^p + \frac{\ell}{2L}\left(\frac{2L+\ell}{t}\right)^p$$

$$\sim (\Delta u)^p \begin{cases} \left(\frac{\ell}{2L}\right)^p & 0 < p < 1 \\ \frac{\ell}{2L} & p \ge 1 \end{cases} \quad \text{for } \ell \ll L$$

where, recall, $\Delta u = 2L/t$.

Thus, $\boxed{\zeta_p = \begin{cases} p & 0 < p < 1 \\ 1 & p \ge 1 \end{cases}}$. Inequality verified, but $\zeta_p = \frac{p}{3}$ only for $p = 3$!

Issue: K41 theory assumes that $h = \frac{1}{3}$ at ~~almost~~ every point of space. For Burgers, this clearly doesn't nec'ly happen. Instead there is one point, $x = 0$, where

$$\delta u(\ell; x) \sim \Delta u \sim \ell^0 \qquad \text{for all } \ell < L$$

and at every other point

$$\delta u(\ell; x) \sim \ell/t \sim \ell^1 \qquad \text{for } \ell \text{ suff small.}$$

This is small scale intermittency: velocity increments are very large in some places and small in others!

Remark: What we learned from this specific Burgers solution is very general for the Burgers equation. Except for fields with initial conditions $\partial_x u_0 > 0$ everywhere, Burgers solutions always develop shocks which dissipate energy in the inviscid limit. Moreover, it is well known that entropy solutions of Burgers (obtained in the inviscid limit) enjoy the regularity

$$u \in BV \cap L^\infty \quad \text{a.e. } t$$

$$\overset{\text{embedding}}{\Rightarrow} \quad u \in B_p^{1/p,\infty} \quad \text{a.e. } t, \quad p \geq 1.$$

For more information, see

W. E, K. Khanin, A. Mazel and Y. Sinai
"Invariant Measures for Burgers Equation with Stochastic Forcing"
Ann. Math. (3): 877-960 (2000).

Review:

J. Bec & K. Khanin, "Burgers Turbulence", Physics Reports 447, 1-66 (2007)