Parisi - Frisch Multifractal Model

Parisi and Frisch (1983) advanced the following heuristic explanation for the anomalous scaling of structure functions.

Assume the local Hölder exponents $h(x)$ obtained in limit $\nu \to 0$ lie in the range $[h_{min}, h_{max}]$, with $0 < h_{min}$ and $h_{max} < 1$.

Define

$$S(h) = \{x : h(x) = h\}$$
$$D(h) = D(S(h)).$$

Since argument is heuristic, we don't specify the dimension, eg $D = D_B, D_H$.

Now, by definition of $S(h)$,

$$|\delta u(r; x)| \sim u_0 \left(\frac{r}{L}\right)^h \quad \text{if} \quad r \le \mathrm{dist}(x, S(h))$$

The fraction of space on which this occurs is estimated as

$$\text{Fraction}\left(x : \mathrm{dist}(x, S(h)) \sim r\right) \sim \left(\frac{r}{L}\right)^{k(h)}$$

where $k(h) = d - D(h)$ is the fractal codimension of $S(h)$.

If the distribution of exponents over space has a smooth weight μ then

$$S_p(r) = \langle |\delta u(r)|^p \rangle$$
$$= \int_{h_{min}}^{h_{max}} d\mu(h) \left[u_0 \left(\frac{r}{L}\right)^h\right]^p \left(\frac{r}{L}\right)^{k(h)}$$
$$\sim u_0^p \int_{h_{min}}^{h_{max}} d\mu(h) \left(\frac{r}{L}\right)^{ph + k(h)} \sim u_0^p \left(\frac{r}{L}\right)^{\zeta_p}$$

by steepest descent, or $\lim_{|r| \to 0} \frac{\ln(S_p(r))}{\ln(r/L)} = \zeta_p$ with

$$\zeta_p = \inf_{h \in [h_{min}, h_{max}]} \{ph + k(h)\}.$$

Example (Burgers)

There are two Hölder exponents

- isolated shocks, $h=0$ with $D(0)=0 \Rightarrow K(0) = 1 - D(0) = 1$
- smooth regions, $h=1$ with $D(1) = 1, \Rightarrow K(1) = 1 - D(1) = 0$

For other $h \neq 0, 1$, $K(h) = +\infty$, formally so that the infimum is taken only over $h=0$ or 1:

$$\zeta_p = \inf_{h \in \{0,1\}} [ph + K(h)] = \inf \{1, p\} = \begin{cases} p & 0 \leq p \\ 1 & p \geq \end{cases}$$

This is the same formula we calculated previously!

Burgers is called bifractal, with just two different exponents.

Note that the relation

$$\zeta_p = \inf_h \{ph + K(h)\}$$

is a Legendre transform. If $K(h)$ is strictly convex ($D(h)$ strictly concave) then the infimum is uniquely achieved for each p at $h_*(p)$ with

$$0 = \frac{\partial}{\partial h}\{ph + K(h)\}\Big|_{h = h_*(p)} = p + K'(h_*(p))$$

$$\Rightarrow \quad K'(h_*(p)) = -p.$$

Now, since $K(h)$ is strictly convex, $p_*(h) := -K'(h)$ is strictly decreasing, so that the inverse function

$$h_*(p) = \inf\{h : p_*(h) < p\}$$

is also decreasing. Then

$$\zeta_p = \inf_h \{ph + K(h)\} = ph_*(p) + K(h_*(p))$$

In this case

$$\frac{d\zeta_p}{dp} = h_*(p) + \cancel{[p + K'(h_*(p))]}^{\,0} \frac{dh_*(p)}{dp} = h_*(p).$$

Thus we find

$$P_*(h) = -\frac{dk}{dh}(h) \quad \text{and} \quad h_*(p) = \frac{d\zeta_p}{dp}.$$

Furthermore,

$$k(h_*(p)) = \zeta_p - p\,h_*(p)$$

$$\Longrightarrow \quad k(h) = \zeta_{p_*(h)} - p_*(h)\,h.$$

Note that $\zeta_p - ph$ is concave in p and

$$\frac{d}{dp}\left[\zeta_p - ph\right] = h_*(p) - h = 0 \quad \text{if} \quad h = h_*(p) \quad \text{or} \quad p = p_*(h).$$

Thus

$$k(h) = \zeta_{p_*(h)} - p_*(h)\,h = \sup_p \{\zeta_p - ph\}.$$

We have argued

$$\zeta_p = \inf_h \{ph + k(h)\}$$

$$k(h) = \sup_p \{\zeta_p - ph\}.$$

This relationship is called <u>Legendre duality</u>. If $k(h)$ is not convex, one can show

$$\bar{k}(h) = \sup_p \{\zeta_p - ph\}$$

is the convex hull of $k(h)$.

Parisi - Frisch (1985) proposed to use these formulae to extract

$$D(h) = \inf_p \{ph + (d - \zeta_p)\},$$

the "<u>multifractal spectrum</u>".

Kestener & Arneodo (2004) obtained $D(h)$ from a 256^3 DNS with $R_\lambda = 140$ and found that the most probable $h_* = h_*(0)$ of about 0.34 ± 0.02, slightly larger than K41 value $h = 1/3$.