L0

Stationary stochastic processes.
(Steady state Mathematics)

1. Definitions : "Weak" vs "strong" stationarity
2. Spectral properties of weakly stationary process.
3. Taylor Diffusion
4. Rmk : x-space vs t-space:

References: Pavliotis, chapter 1.
Frisch, "Legacy", chapter 4 ["Stochastic process" vs "Random function"].

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① Definitions

Def [S.S].

Let $(X_t, t \geq 0)$ or more generally $(X_t, t \in T)$ a stochastic process.

X_t is said to be a strictly stationary

iff

$\forall k \in \mathbb{N} \quad \forall (t_1 \ldots t_k) \in T \quad \forall s$ such that $(t_1+s \ldots t_k+s) \in T$.

$$\left(X(t_1) \ldots X(t_k) \right) \overset{Law}{\sim} \left(X(t_1+s) \ldots X(t_k+s) \right).$$

Examples:

① Let $Y_0 \ldots Y_n \sim Y$ iid. (real valued R.V)

Define $X : \mathbb{N} \times \Omega \longrightarrow \mathbb{R}$ such that $X(i) = Y_i$.

Then $T = \mathbb{N}$ and $(X_t, t \in \mathbb{N})$ is S.S.

Indeed: $P[\forall i \ X(t_i) \in \Delta x_i] = \prod_i P(Y \in \Delta x_i) = P[\forall i \ X(t_i+s) \in \Delta x_i]$

② Let $(X_t, t \in \mathbb{N})$ such that $X_t = Y_0$.
then $(X_t, t \in \mathbb{N})$ is S.S. $P[\forall i \ X(t_i) \in \Delta x_i] = P[Y_0 \in \bigcap_i \Delta x_i]$

Def [w.s].

$(X_t, t\in T)$ such that $\langle X_t^2\rangle < +\infty$ is weakly stationary

iff

(i) $\langle X_t\rangle = \text{cste} = \mu$

(ii) $\exists\, \mathcal{C} : T \to \mathbb{R}$ $\forall (s,t)\in T^2$ $\langle (X(t)-\mu)(X(s)-\mu)\rangle = \mathcal{C}(t-s)$

$\mathcal{C}$ is the auto-correlation of the process.

Rmks: ① (s.s) $\Longrightarrow$ (w.s).

Indeed: (i) $X(t_1) \sim X(t_2)$ $\Rightarrow$ $\mu(t_1) = \mu(t_2) = \mu$ $\forall t_1 t_2$.

(ii) $\langle (X_t-\mu)(X_s-\mu)\rangle \underset{\uparrow\, n=2}{=} \langle (X_{t-s}-\mu)(X_0-\mu)\rangle = \mathcal{C}(t-s)$

② (w.s) $\not\Rightarrow$ (s.s).

Example: $X_t : \mathbb{N}\times\Omega \to \mathbb{R}$ $\begin{cases} X_{2t} = Y_t \\ X_{2t+1} = Z_t \end{cases}$

where: $Y_0 \dots Y_n$, $Z_0 \dots Z_n$ are indep^t iid.

$Y_i \sim Y$ $\quad p_Y(x) = \frac{1}{2}\delta(x) + \frac{1}{4}\delta(x-1) + \frac{1}{4}\delta(x+1)$

$p_Z(x) = \frac{7}{8}\delta(x) + \frac{1}{16}\delta(x-2) + \frac{1}{16}\delta(x+2)$

$\longrightarrow$ $\langle X_t\rangle = 0$.

$\langle X_t X_s\rangle = \delta(t-s)\,\frac{1}{2}$.

$\langle X_t^4\rangle = \begin{cases} \frac{1}{2} & \text{if } t = 0\,[2]. \\ \frac{2^4}{16} + \frac{2^4}{16} = 1 \text{ ?} & \text{if } t = 1\,[2]\ ✓ \end{cases}$ $\Rightarrow \langle X_t^4\rangle \neq$ cste.

(w.s) is a statement about the "low order statistics" of the stochastic process.

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Properties of (W.S) processes:

Let $(X_t, t\in\mathbb{R})$ be a (W.S) stochastic process.

such that $\begin{cases} \langle X_t\rangle = 0 \\ \mathcal{C}(t-s) \doteq \langle X_t X_s\rangle \end{cases}$

[Sometimes we see the denomination centered stochastic process for a (W.S) s.p with mean zero].

Then

- $\mathcal{C}$ is non-negative definite:

$$\forall c_1 \dots c_n \in \mathbb{C}^n\ \forall t_1 \dots t_n \quad \sum_{i,j} c(t_i - t_j)\, c_i c_j^* \geq 0 .$$

Indeed: $\sum_{i,j} \mathcal{C}(t_i - t_j)\, c_i c_j^* = \langle \sum_{i,j} X(t_i) c_i\ X(t_j) c_j \rangle = \langle |\sum_i X(t_i) c_i|^2 \rangle$

- $\mathcal{C}$ is symmetric $\quad \mathcal{C}(z) = \mathcal{C}(-z)$ ✓

- $\mathcal{C}(0) = \sup_z |\mathcal{C}(z)|$

Indeed: $|\mathcal{C}(z)| = |\langle X(z) X(0)\rangle| \leq \langle X^2(z)\rangle^{1/2} \langle X^2(0)\rangle^{1/2} = \mathcal{C}(0)$

(optional) • Continuity of $\mathcal{C}$ is equivalent to continuity of X_t in L^2 sense.

$$|c(t+h) - c(t)|^2 = |\langle X(t+h) X(0) - X(t) X(0)\rangle|^2$$
$$= |\langle X(0)^* [X(t+h) - X(t)]\rangle|^2$$

$$\leq \langle X(0)^2\rangle \langle (X(t+h) - X(t))^2\rangle . \quad (*)$$
$$= 2\langle X(0)^2\rangle [c(0)^* - c(h)] .$$
$$= 2\, \mathcal{C}(0)\, [c(0) - c(h)] .$$

($\Rightarrow$) $\mathcal{C}$ continuous $\Rightarrow$ $\mathcal{C}$ C^0 at 0 $\Rightarrow$ X C^0 in L^2 from $\overset{(*)}{\downarrow}$.

($\Leftarrow$) X C^0 in L^2 $\Rightarrow$ c C^0 from $\overset{\uparrow}{(*)}$

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- If X_t is gaussian, then it is also strongly stationary

$\mathcal{D}/(t_1 - t_2)$. $\exists \mu \, (X_{t_1} - X_{t_2}) \sim \mathcal{N}(\mu, C)$

observe: $\langle X_{t_i} X_{t_j} \rangle = C_{ij} = \langle X_{t_i+s} X_{t_j+s} \rangle$
$\uparrow (t, s)$

$\langle X_i \rangle = \mu_i = \langle X_{t+s} \rangle$.

$\Longrightarrow (X_{t_1} - X_{t_2}) \overset{\mathcal{L}}{\sim} (X_{t_1+s} \ldots X_{t_2+s}) \sim \mathcal{N}(\mu, C)$

Def [Correlation time]

Let $(X_t, t \in \mathbb{R})$ a weakly stationary s.p with C correlation function,

e.g: $\int_0^\infty |\mathcal{C}| < +\infty$

Then the correlation time, is

$$\tau_{corr} \hat{=} \frac{\int_0^\infty \mathcal{C}(\tau) d\tau}{\mathcal{C}(0)}$$

Example. ① Gaussian process with correlation function:

$\mathcal{C}(t) = \mathcal{C}(0) e^{-|t|/\tau}$ is stationary.

it is the stationary "Ornstein-Uhlenbeck" process.

with correlation time $\tau_c = \int_0^\infty e^{-t/\tau} dt = \ldots \tau$ (!)

② $(B.M)$ is not gaussian: $\langle X_t X_s \rangle = \min(s,t)$.

Yet the increments of $(B.M)$ are stationary

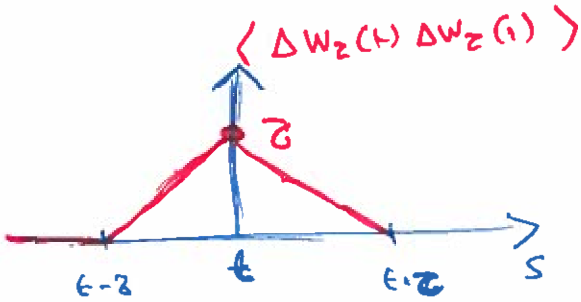
Specifically, define:

$$\Delta W_\tau(t) \doteq [W(t+\tau) - W(t)], \quad \text{for } \tau > 0.$$

Then

$$\langle \Delta W_\tau(t)\, \Delta W_\tau(s) \rangle = \langle W(t+\tau)W(s+\tau) - W(t+\tau)W(s) + W(t)W(s+\tau)\times(-1) + W(t)W(s) \rangle$$

$$= [t \wedge s] + \tau - (t+\tau)\wedge s - t\wedge(s+\tau) + \ldots$$



$$= \tau - s + s - \tau + 2s = 0 \quad \text{if} \quad s+\tau < t$$

$$= s + \tau - s - t + s = \tau + s - t \quad \text{if} \quad t-\tau < s < t.$$

$$= \tau. \quad \text{if} \quad s = t.$$

Explicitly:
$$\langle \Delta W_\tau(t)\, \Delta W_\tau(s) \rangle = \begin{cases} 0 & \text{if } |t-s| > \tau. \\ \tau - |s-t| & \text{if } |s-t| < \tau. \end{cases}$$

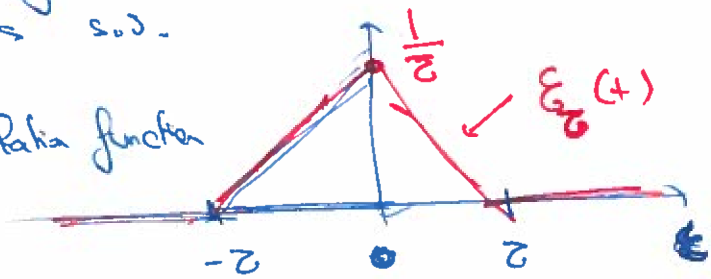
③ Let us define:

$$y_\tau(t) \doteq \frac{\Delta W_\tau(t)}{\tau}$$

Then
- $y_\tau(t)$ has gaussian statistics and is w.s.

$\Longrightarrow$ It is s.s.

- $y_\tau(t)$ has correlation function



Observe:
- $\mathcal{E}_\tau(0) = \frac{1}{\tau} \xrightarrow[\tau\to\infty]{} \infty$

- $\int_0^\infty \mathcal{E}_\tau(t)\, dt = \frac{1}{\tau} \times \tau = 1.$

In this sense: "$\mathcal{E}_\tau \longrightarrow \delta$"

Although the identity "$y(t) = \dfrac{dW}{dt}$" is ill-defined (Remember: W is nowhere differentiable!),

$\lim_{\tau \to 0} y_\tau(t)$ can be taken as a defining identity for <u>white noise</u>. To justify the name, we need to introduce the concept of <u>power spectrum</u>.

② <u>Spectral properties of w.s. stochastic process.</u>

<u>Def</u>: [Spectra].

The spectrum of a $\overline{\text{centered}}$ (w.s) s.p $(X_t, t \in \mathbb{R})$ with correlation function $\mathcal{E} : \mathbb{R} \to \mathbb{R}$ such that $\int_0^\infty |\mathcal{E}| < +\infty$.

is defined as:

$$f[\omega] \stackrel{\circ}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\tau\omega} \mathcal{E}[\tau] \, d\tau.$$

$$= \frac{1}{\pi} \int_0^\infty \cos \omega\tau \; \mathcal{E}[\tau] \, d\tau.$$

<u>Note</u> $f[\omega] = f[-\omega]$. and $f[\omega] \geq 0$. [Wiener-Khinchin]

<u>Rmk</u>: • <u>Conversely</u>: $\mathcal{E}[\tau] = \int_{-\infty}^{\infty} e^{i\tau\omega} f[\omega] \, d\omega$.

• $\mathcal{E}[0] = \langle X_t^2 \rangle = \int_{-\infty}^{\infty} f[\omega] \, d\omega = 2 \int_0^\infty \overbrace{f(\omega)\, d\omega}^{\text{contribution of fq } \in [\omega, \omega+d\omega] \text{ to } \langle x^2 \rangle}.$

Rationale: [Not a proof!].

cf [Frisch, Chap 3].

- Assume one may formally decompose the s.p $(X_t, t \in \mathbb{R})_{(1)}$:

$$X(t,\omega) = \int_{-\infty}^{\infty} df\, \hat{X}[f,\omega]\, e^{ift}.$$

ω: Realization

So that $$\hat{X}[f,\omega] = \frac{1}{2\pi} \int_{\infty}^{\infty} e^{-ift} X(t,\omega)\, dt.$$

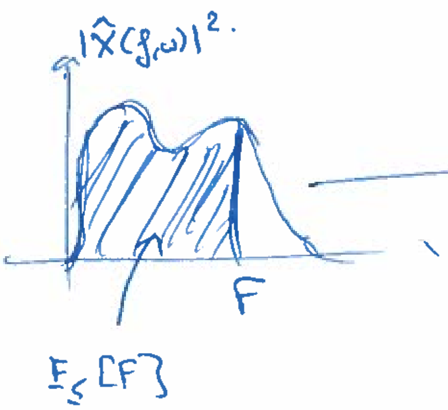
[Random Fourier coef].

- Define the low-pass signal:

$$X_{\leq F}(t,\omega) = \int_{-\infty}^{\infty} \mathbb{1}_{f \leq F}\, e^{ift}\, \hat{X}[f,\omega]\, df.$$

Then $\langle |X_{\leq F}(t,\omega)|^2 \rangle \doteq 2\, E_{\leq}[F]$ "Energy contained in the low frequency"

2: Conventional



(Not an elephant!)

- Now define: $e(f) \doteq \dfrac{dE_{\leq}[f]}{df} \Rightarrow dE_{\leq}[f] = e(f)\, df.$

Nec: $E_{\leq} \nearrow \Rightarrow \boxed{e(f) \geq 0}$.

obtain from : lhs: $\langle |X_{\leq F}|^2 \rangle \to \langle X^2 \rangle$.

rhs: $2E[F] \to 2E[\infty] = 2\int_0^{\infty} e(f)\, df$

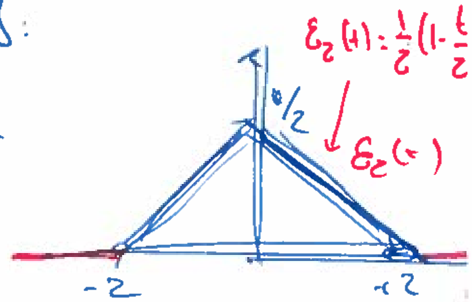
Examples:

• O.U : $\mathcal{E}(\tau) = \langle V_0^2\rangle e^{-\alpha|\tau|}$

$$S(\omega) = \frac{\mathcal{E}_0}{\pi}\,\mathrm{Re}\int_0^\infty e^{i\omega\tau-\alpha\tau}\,d\tau = \mathrm{Re}\,\frac{\mathcal{E}_0}{\pi}\,\frac{(-1)(i\omega+\alpha)}{(i\omega-\alpha)(i\omega+\alpha)}$$

$$= \frac{\mathcal{E}_0}{\pi}\,\frac{\alpha}{\omega^2+\alpha^2}.$$

• Increments of B.M [Regularized white noise]:



$$S[\omega] = \frac{1}{\pi}\,\mathrm{Re}\int_0^\infty e^{i\omega t}\,\mathcal{E}_\tau(t)\,dt$$

$$= \frac{\mathrm{Re}}{\pi}\int_0^\tau e^{i\omega t}\,\frac{1}{\tau}\left(1-\frac{t}{\tau}\right)dt.$$

$$\underset{\text{I.P.P.}}{=} \frac{\mathrm{Re}}{\pi\tau}\left[\underbrace{\left[\frac{1}{i\omega}e^{i\omega t}\left(1-\frac{t}{\tau}\right)\right]_0^\tau}_{\mathrm{Re}(\cdot)=0} + \frac{1}{\tau i\omega}\int_0^\tau e^{i\omega t}\,dt\right]$$

$$\boxed{S_\tau(\omega) = \frac{1}{\pi\tau^2\omega^2}\left(1-\cos\omega\tau\right)}. \quad (\geq 0\ !)$$

$$\underset{\tau\to0}{=} \frac{1}{\omega^2\pi\tau^2}\,\frac{\omega^2\tau^2}{2} + o(1)$$

$$\boxed{S_\tau(\omega) \xrightarrow[\tau\to0]{} \frac{1}{2\pi}}$$

Every frequency contributes equally ⟹ the noise is "white"!.

By contrast, the O.U is a "red noise"

Rmk:

As $\alpha \to \infty$ (ie $\tau_{corr} \to 0$), the O.U spectrum also becomes flat.

Upon Rescaling $\mathcal{E}_0 \sim \frac{\alpha}{2}$, we formally obtain:

$$S_\alpha(\omega) \sim \frac{\cancel{\alpha}}{2\pi} \frac{\alpha}{\cancel{\alpha^2}} = \frac{1}{2\pi}$$

ie the "short-term memory" O.U. behaves as white noise.

Exercises:

① • Show that the increments of a (w.s) process are also (w.s): ie $\Delta_\tau X \doteq (X(t+\tau) - X(t), t \in \mathbb{R})$ is (w.s).
• Define their correlation function as $\mathcal{E}_{\Delta_\tau}$

Show $\mathcal{E}_{\Delta_\tau}(0) = 2\left[\mathcal{E}(0) - \mathcal{E}(\tau)\right]$.

$$= 2 \int_{-\infty}^{\infty} S(\omega)\left(1 - e^{i\omega\tau}\right) d\omega$$

② Show that no (s.p) that is (w.s) exist that has concomitantly:

$S(\omega) \sim \omega^{-4}$ & $\mathcal{E}(0) < \infty$.

③ Show that the increments of of (w.s) s.p. can have (IR) (UV) ... behavior $S(\omega) \sim \omega^{-n}$ provided $1 < n <$ ...

③ Taylor Diffusion

A first consequence of (w.s). (optionnel).

Thm [Ergodicity].

$(X_t, t\in\mathbb{R})$ centered (w,s) s.p with L^1 correlation function:

$$D \doteq \int_0^\infty C(\tau)\, d\tau .$$

Then

$$\left\langle \left| \frac{1}{T}\int_0^T X(s)\, ds \right|^2 \right\rangle \xrightarrow[T\to\infty]{} 0 .$$

a/ Direct calculation:

$$\left\langle \left| \frac{1}{T}\int_0^T x\, ds \right|^2 \right\rangle \underset{\text{sot}}{=} \frac{2}{T^2}\int_0^T dt \int_0^t d\tau\, C(\tau)$$

$$\underset{\text{IPP}}{=} \frac{2}{T^2}\left[T\int_0^T d\tau\, C(\tau) - \int_0^T dt\, C(t)\, t \right]$$

$$= \frac{2}{T}\left[\int_0^T dt \left(1-\frac{t}{T}\right) C(t) \right]$$

$$\leq \frac{2}{T}\int_0^T C \leq \frac{2D}{T} \longrightarrow 0 .$$

[11]

Phenomenon [Taylor Diffusion].

Consider $(v(t), t \in \mathbb{R})$ a centred (s.p), (w.s) with $\int_0^\infty |\mathcal{C}| < \infty$

$$D \doteq \int_0^\infty \mathcal{C}.$$

Define the integrated (s.p) $X(t, \omega) \doteq \int_0^t v(s,\omega)\, ds$. $\quad C^0$.

Then

$$\frac{d}{dt} \langle X^2 \rangle_t \xrightarrow[t \to \infty]{} 2D.$$

i.e.

$$\langle X^2 \rangle \underset{t \to \infty}{\sim} 2Dt.$$

D/ $\frac{d}{dt}\langle x^2 \rangle = 2 \langle v(t) \int_0^t v(s)\, ds \rangle \to 2D$.

$\Longrightarrow$ Asymptotic emergence of a diffusive behavior!

Explicit example:

Again, (O.U) $\mathcal{C}(\tau) = \langle v_0^2 \rangle e^{-\alpha \tau}$.

Then

$$\frac{d}{dt}\langle X^2 \rangle = 2 \langle v_0^2 \rangle \int_0^t e^{-\alpha \tau}\, d\tau \qquad \text{(exact)}$$

$$= 2 \langle v_0^2 \rangle \frac{1}{\alpha} (1 - e^{-\alpha t}).$$

$$\xrightarrow[\alpha t \gg 1]{} 2D \qquad D = \frac{\langle v_0^2 \rangle}{\alpha} = \tau \langle v_0^2 \rangle$$

$[D] = m^2 s^{-2} s = m^2 s^{-1}$

$$\xrightarrow[\alpha t \ll 1]{} 2 \langle v_0^2 \rangle / \cancel{\alpha}\, \cancel{\alpha} t \Longrightarrow \text{“ballistic behaviour”}$$

Further Examples:

See the notebook "Taylor-Diffusion" which describes emergence of diffusive behavior from deterministic dynamics.

① Standard Map:

$$\begin{cases} J[t+1] = J[t] + K\sin\theta \\ \theta[t+1] = \theta[t] + J[t+1] \end{cases}$$

Under "Random Phase" approx $\langle \sin\theta(t)\sin\theta(t')\rangle = \delta(t-t')\langle \sin^2\theta\rangle$.

Show that: $\langle J\rangle = 0$.

$\langle J^2\rangle_T \sim 2DT$ with $D = \frac{K^2}{4}$.

$(T\to\infty)$.

② Particle driven by OU.

③ 1 point dispersion in a "turbulent signal"

(4) Rmk: From t-space to x-space.

[Teaser].

One may as well think of the parametric variable as a space variable.

In which case, terminology of stat. process translate into:

t - space	x - space
Stationarity	homogeneity.
Frequency f.	wave number k.
Power spectrum	Energy spectrum.
high f	high k, "small scales."

This is particularly relevant to describe turbulent velocity fields!