Illustration of Liouville theorem:

Consider Orszag's "poor man turbulence model"

(cf Orszag, 1977):

$$(*) \begin{cases} \dot{z}_i = F_i[z] \doteq z_{i+1} z_{i+2} + z_{i-1} z_{i-2} - 2 z_{i-1} z_{i+1} \\ z_i(t=0) = z_i(0) \in \mathbb{R} \qquad i \in [\![0; N-1]\!], \ z_{i+N} = z_i \end{cases}$$

① * $(*)$ is obviously a conservative dynamics, it even satisfies a detailed Liouville theorem:

i.e. $\forall i \in [\![0, N-1]\!] \quad \dfrac{\partial \dot{z}_i}{\partial z_i} = \dfrac{\partial}{\partial z_i} F_i = 0.$

② * $(*)$ conserves the energy $E[z] \doteq \sum_{i=0}^{N-1} z_i^2$.

③ * the Liouville operator is computed as $\boxed{\mathcal{L} A = \sum_{i=0}^{N-1} F_i \dfrac{\partial}{\partial z_i} A}$

i.e. $\mathcal{L} A = \sum_{i=0}^{N-1} \left(z_{i+1} z_{i+2} + z_{i-1} z_{i-2} - 2 z_{i-1} z_{i+1} \right) \dfrac{\partial}{\partial z_i}$

④ * the adjoint operator $\mathcal{L}^*$ such that $\partial_t \rho = \mathcal{L}^* \rho$ is

$$\mathcal{L}^* = - \sum_{i=0}^{N-1} \frac{\partial}{\partial z_i} F_i = - \sum_{i=0}^{N-1} F_i \frac{\partial}{\partial z_i} = - \mathcal{L}.$$

⑤ * As a result: from the observation $\mathcal{L} E[z] = 0$,

we get that ~~any~~ measure $\rho_\varphi \doteq \varphi[E]$

for φ sufficiently smooth $\varphi : \mathbb{R} \to \mathbb{R}^+$ is invariant;

⑥ . Consider in particular the family of "gibbsian" measures:

$$\rho_\beta = \frac{1}{Z_\beta} \exp - \beta E . \qquad Z_\beta = \prod_{i=0}^{N-1} \sqrt{\frac{\pi}{\beta}} .$$

$$= \prod_{i=1}^{N-1} \left(\exp - \beta x_i^2\right) \sqrt{\frac{\pi}{\beta}} .$$

Then ρ_β is an invariant measure.

If $z(t_0) \sim \rho_\beta$ then $\forall t \geq 0 \quad \langle O(z_t) \rangle_\beta = \langle O(z_0) \rangle_\beta$

For example: $\forall i \in [0; N-1] \quad \langle z_i \rangle = 0 .$

$$\langle z_i^2 \rangle = \frac{1}{2\beta} \qquad \& \quad \langle E \rangle = \frac{N}{2\beta} ,$$

⑦ This is not the case for the pseudo gibbsian measure:

$$\tilde{\rho}_\beta = \prod_{i=1}^{N-1} \exp - \underline{\underline{\beta_i}} \, x_i^2 \sqrt{\frac{\pi}{\beta_i}} . \qquad \text{with } \beta_i \neq \beta_j .$$

$$\Longrightarrow \mathcal{L}^* \tilde{\rho}_\beta = - \sum_{i=0}^{N-1} F_i \frac{\partial}{\partial z_i} \left[\prod_\ell e^{-\beta_\ell z_\ell^2} \sqrt{\frac{\pi}{\beta_\ell}} \right]$$

$$= - \sum_{i=0}^{N-1} F_i \; 2 z_i \beta_i \, \tilde{\rho}_\beta ,$$

$$= -2 \tilde{\rho}_\beta \sum_{i=0}^{N-1} \left(z_i z_{i+1} z_{i+2} \beta_i + \beta_i z_i z_{i-1} z_{i-2} - 2 \beta_i z_{i-1} z_i z_{i+1} \right)$$

$$\boxed{\mathcal{L}^* \tilde{\rho}_\beta = -2 \tilde{\rho}_\beta \sum_{i=0}^{N-1} z_i z_{i+1} z_{i+2} \left[\beta_i + \beta_{i+2} - 2 \beta_{i+1} \right]}$$

$\neq 0$ (in general)

for instance such:

- Liouville thm connects conservation laws to specific classes of invariant measures, prescribed by the dynamical invariants.

- In the context of turbulence, this has justified formal statistical mechanics computation of both microcanonical & canonical measures. This approach has been particularly fruitful in 2D where the presence of 2 invariants produce non-trivial predictions & inequivalence of ensembles, that describe spontaneous organization towards large scales .. ("Bose - Einstein Condensate")

cf { Kraichnan & Montgomery, 1980 for example.
Robert & Sommeria, 1990
Miller, Weichman, Cross, 1992.

- In 3D, invariant measures based on E are however trivial:

As in the case of Onsager's model: Requiring $\langle E \rangle_{n \to \infty} = O(1)$

produce $\langle E(k) \rangle \underset{k \to \infty}{\longrightarrow} 0$. Hence equipartition spectrum is

degenerate, as it produces either infinite or trivial vanishing

energy state!