Cauchy invariants and Cauchy-Lagrangian numerical methods

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Lecture I: Cauchy's almost forgotten 1815 work

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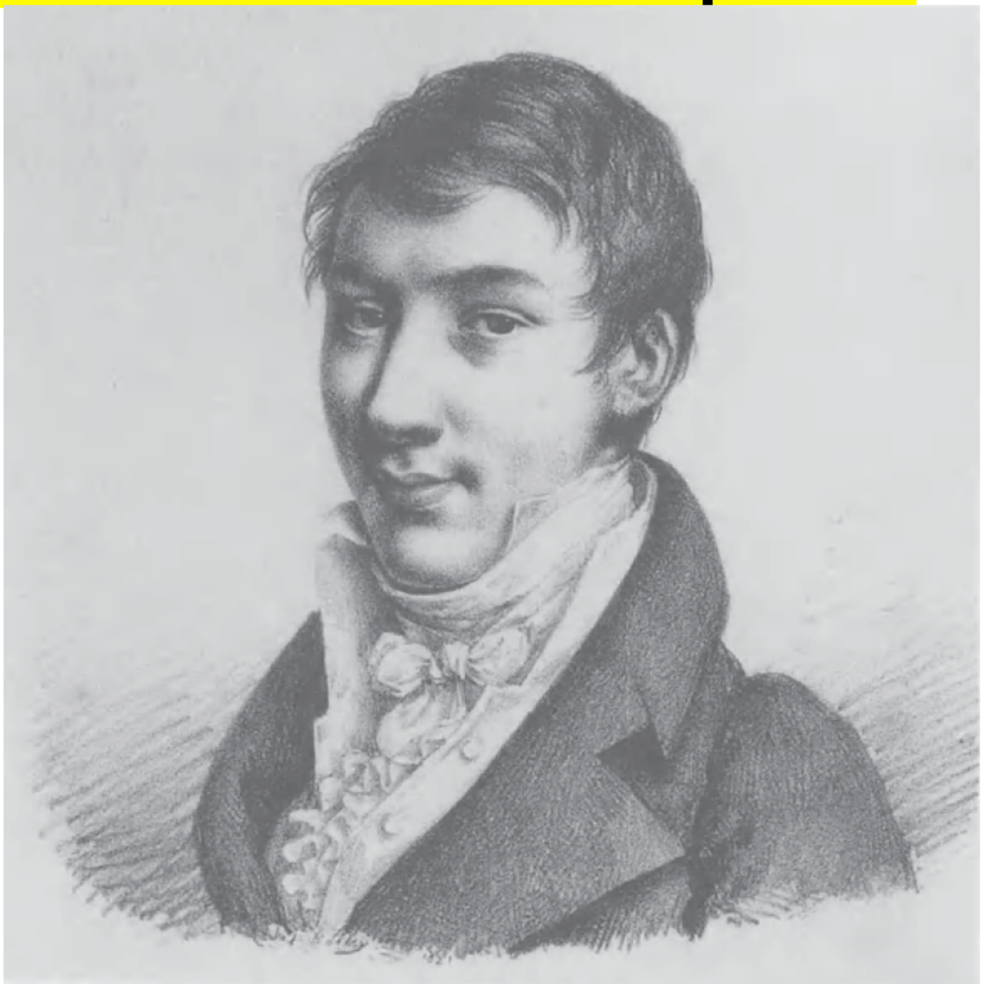
Cauchy and the 1815 "Grand prix de mathématique"



- Augustin-Louis Cauchy 1789-1857
- Engineering studies in Paris
- Junior engineer in Cherbourg 1810
- Engineer Ourcq canal (Paris) 1813
- Significant mathematical production started early: in 1813 the "geometry" Section of the Academy ranked him 2nd for election as member
- March 1816: Cauchy appointed by the King member of the Academy
- December 1813, the Paris Academy decides that its1815 "Great mathematics prize" will be on: *the problem of waves on the surface of a liquid of arbitrary depth*
- December 26, 1815 Cauchy obtains the prize for a three-part memoir, eventually published (310 pages) in 1827 in *Mémoires des savans étrangers* (here, only the 2nd part is of interest)

Eulerian and Lagrangian coordinates

- In 1747 the Berlin Academy set a prize "*to determine ... at all times the speed and direction of the wind in all places.*" In modern notation, to find the Eulerian velocity $\boldsymbol{v}(\boldsymbol{x},\, t)$.
- In 1757 Leonhard Euler wrote the dynamical evolution equations for a 3D incompressible ideal (inviscid) fluid

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p, \qquad \nabla\cdot\boldsymbol{v} = 0$$

- Lagrange's 1788 variational formulation of the Euler equations made use of the map $\boldsymbol{a}\mapsto\boldsymbol{x}(\boldsymbol{a},\, t)$ of the initial position $\boldsymbol{a}$ of a fluid particle to its current position $\boldsymbol{x}$, solution of the characteristic equation $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x},\, t), \quad \boldsymbol{x}(\boldsymbol{a}, 0) = \boldsymbol{a}\,.$
- He obtained the following Lagrangian dynamical equations

$$\sum_{k=1}^{3} \ddot{x}_k \nabla^{\mathrm{L}} x_k = -\nabla^{\mathrm{L}} p, \quad \det\left(\nabla^{\mathrm{L}}\boldsymbol{x}\right) = 1$$

where $\nabla^{\mathrm{L}}\boldsymbol{x} := \nabla_{\boldsymbol{a}}\boldsymbol{x}$ is the Jacobian matrix of the map.

The Cauchy invariants equations

Cauchy takes the Lagrangian curl of Lagrange's equation

$$\nabla^{\mathrm{L}} \times \textstyle\sum_{k=1}^{3} \ddot{x}_k \nabla^{\mathrm{L}} x_k = 0$$

which he then integrates in time, to obtain *the Cauchy invariants equations*

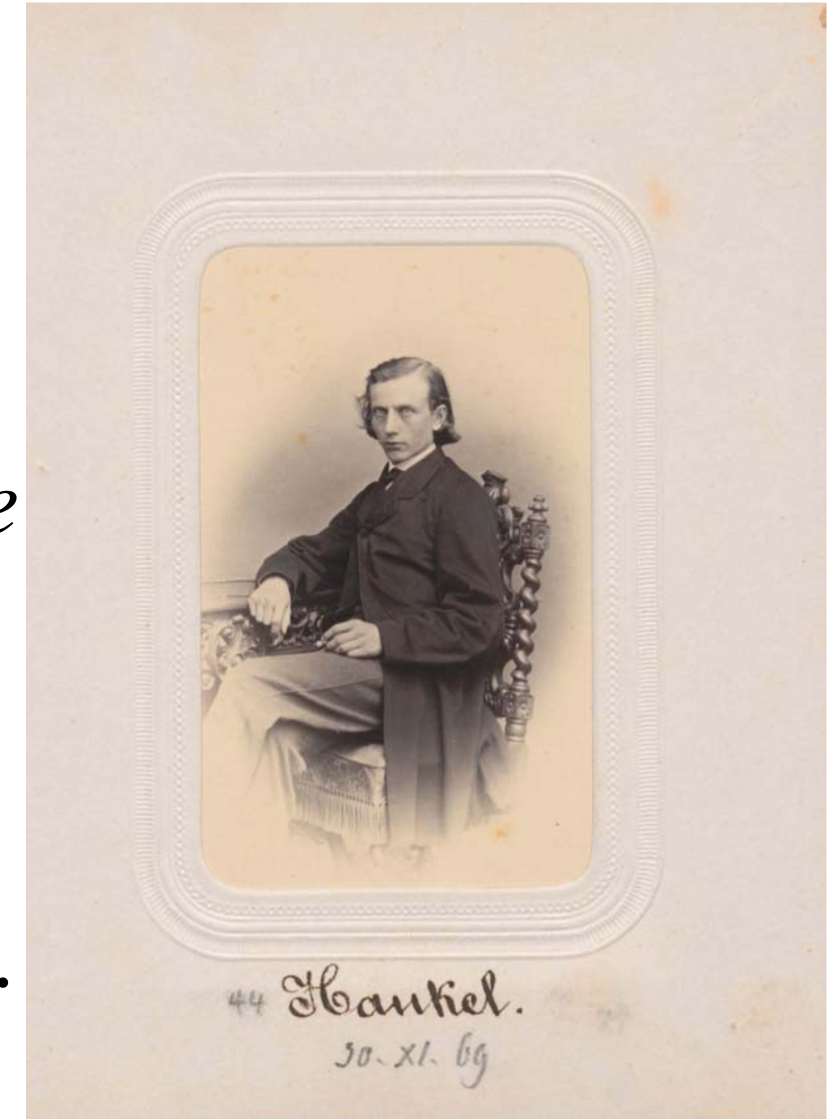
$$\sum_{k=1}^{3} \nabla^{\mathrm{L}} \dot{x}_k \times \nabla^{\mathrm{L}} x_k = \nabla^{\mathrm{L}} \times \sum_{k=1}^{3} \dot{x}_k \nabla^{\mathrm{L}} x_k = \boldsymbol{\omega}_0\,,$$

where $\boldsymbol{\omega}_0 = \nabla^{\mathrm{L}} \times \boldsymbol{v}_0$ is the initial vorticity

$$(15)\left\{\begin{aligned}
&\frac{du}{db}\frac{dx}{da} - \frac{du}{da}\frac{dx}{db} + \frac{dv}{db}\frac{dy}{da} - \frac{dv}{da}\frac{dy}{db} + \frac{dw}{db}\frac{dz}{da} - \frac{dw}{da}\frac{dz}{db} = \frac{du_0}{db} - \frac{dv_0}{da},\\
&\frac{du}{dc}\frac{dx}{da} - \frac{du}{da}\frac{dx}{dc} + \frac{dv}{dc}\frac{dy}{da} - \frac{dv}{da}\frac{dy}{dc} + \frac{dw}{dc}\frac{dz}{da} - \frac{dw}{da}\frac{dz}{dc} = \frac{du_0}{dc} - \frac{dw_0}{da},\\
&\frac{du}{dc}\frac{dx}{db} - \frac{du}{db}\frac{dx}{dc} + \frac{dv}{dc}\frac{dy}{db} - \frac{dv}{db}\frac{dy}{dc} + \frac{dw}{dc}\frac{dz}{db} - \frac{dw}{db}\frac{dz}{dc} = \frac{dv_0}{dc} - \frac{dw_0}{db}.
\end{aligned}\right.$$

Telles sont les intégrales que nous avions annoncées. Si l'on

Hermann Hankel’s 1861 Göttingen Preisschrift



- Hermann Hankel, 1839-1873, student of Moebius, Riemann, Weierstrass, Kronecker
- Known for Hankel matrices, transforms and functions
- Also historian of mathematics: *Zur Geschichte der Mathematik in Alterthum und Mittelalter*
- In 1858 Helmholtz published an important paper on vortex dynamics in which he gave a somewhat heuristic derivation of the Lagrang. invariance of the flux of the vorticity through an infinitesimal piece of surface
- In 1860 Göttingen University set up a prize: *The general equations for determining fluids motions may be given in two ways, one of which is due to Euler, the other one to Lagrange. The illustrious* ***Dirichlet*** *pointed out in the posthumous paper “On a problem of hydrodynamics" the hitherto almost completely neglected advantages of the Lagrangian approach, but he seems to have been prevented, by a fatal disease, from a deeper development thereof. So, this Faculty asks for a theory fluids based on the equations of Lagrange, yielding, at least, the laws of vortex motion discovered otherwise by the illustrious* ***Helmholtz****.*

Hankel's proof of the Helmholtz and circulation theorems

Hankel uses Cauchy's 1815 invariants equations, in the form

$$\nabla^{\mathrm{L}} \times \textstyle\sum_{k=1}^{3} \dot{x}_k \nabla^{\mathrm{L}} x_k = \boldsymbol{\omega}_0$$

In the Lagrangian space of initial fluid positions he takes a finite piece of smooth surface S_0 limited by a contour C_0 and their images by the Lagrangian map from 0 to t, S and C, respectively.

He then applies the *Kelvin*-Stokes-*Hankel* theorem at time zero and at time t, to obtain

circulation

$$\int_{C_0} \boldsymbol{v}_0 \cdot d\boldsymbol{a} = \int_{S_0} \boldsymbol{\omega}_0 \cdot \boldsymbol{n}_0 \, d\sigma_0 = \int_{C_0} \Sigma_k v_k \nabla^{\mathrm{L}} x_k \cdot d\boldsymbol{a} = \int_C \boldsymbol{v} \cdot d\boldsymbol{x} = \int_S \boldsymbol{\omega} \cdot \boldsymbol{n} \, d\sigma$$

vorticity flux

where $\boldsymbol{\omega} := \nabla \times \boldsymbol{v}$ and use has been made of $\displaystyle\sum_{k=1}^{3} v_k \nabla^{\mathrm{L}} x_k \cdot d\boldsymbol{a} = \boldsymbol{v} \cdot d\boldsymbol{x}$

Hankel has thus not only proved Helmholtz's theorem, but obtained an *integral invariant* (circulation theorem), which states that:

$$\int_{C_0} \boldsymbol{v}_0 \cdot d\boldsymbol{a} = \int_C \boldsymbol{v} \cdot d\boldsymbol{x} \, .$$

20th century Cauchy invariants rediscovered and finally credited to Cauchy

- Eckart (1960) rederived the Cauchy invariants (without attribution) and then, in 1963 observed in an incidental way, that these invariants follow from the relabeling symmetry. Almost certainly, he meant "by use of Noether's (1918) theorem."
- Newcomb (1967) explicitly applied Noether's theorem to obtain the Cauchy invariants, again without attribution. Then several authors did the same; such work was reviewed by Salmon (1988).
- Abrashkin, Zen'kovich and Yakubovich (1996) and Zakharov and Kuznetsov (1997) made the correct attribution to Cauchy. Actually, it seems that, among Russian scientists, the expression "Cauchy invariants," as they are now called, was used during most of the nineties.
- Besse and Frisch (2017) showed that Cauchy invariants equations are obtained for any exact Lie-advected form (the vorticity 2-form is an instance). They also showed that Cauchy's well-known vorticity formula $\omega = \omega_0 \cdot \nabla^{\mathrm{L}} x$, follows by Hodge duality.

Cauchy invariants and Cauchy-Lagrangian numerical methods

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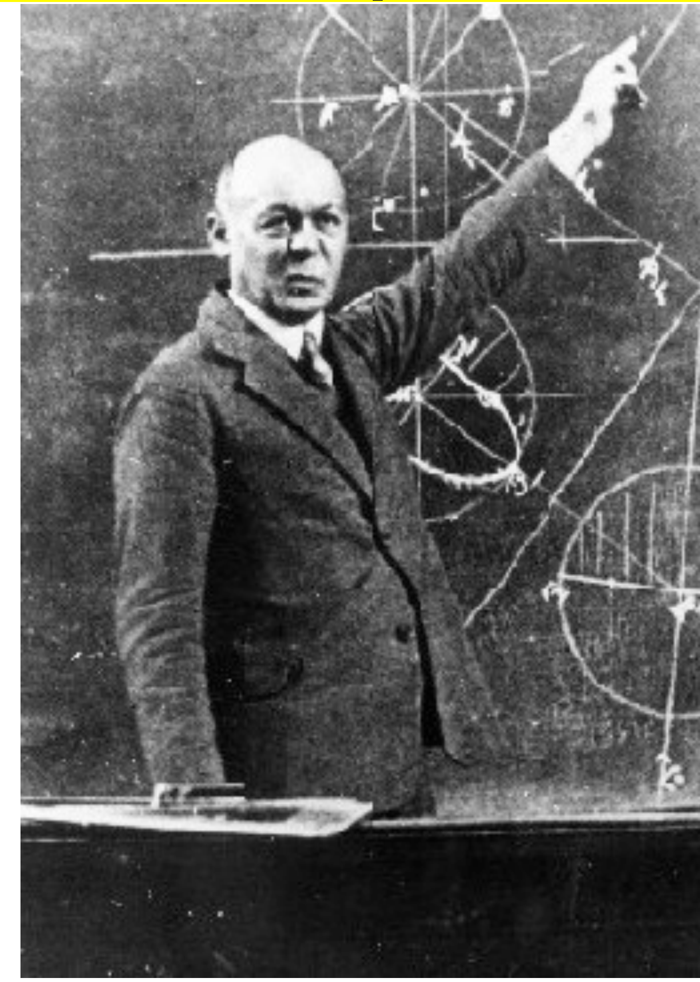
Lecture II: Time-analyticity of the Lagrangian map

The birth of functional analysis for the Euler equations

Witold Wolibner
1902-1961
(Warsaw Polytech. Inst.)

The first proof of persistence for some time of initial regularity for the solutions of the 3D Euler equations was given by Lichtenstein (1927) using tools introduced by Hölder. This effort was continued by Witold Wolibner, Ernst Hölder, who proved all-time regularity for 2D in 1933 and many others.



Leon Lichtenstein 1878-1933
(U. Leipzig: 1922-1933)

Lichtenstein assumed that the initial vorticity satisfies an "H-condition", i.e. is Hölder continuous. For space-periodic solutions to the 3D Euler equations, Lichtenstein's key estimate reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{\omega}(t)|_\alpha < C|\boldsymbol{\omega}(t)|_\alpha^2 \qquad 0<\alpha<1, \quad C>0 \qquad |\boldsymbol{\omega}|_\alpha := \max_{\boldsymbol{x}\in\mathbb{T}^3}|\boldsymbol{\omega}(\boldsymbol{x})| + \sup_{\boldsymbol{x}_1,\,\boldsymbol{x}_2\in\mathbb{T}^3,\,\boldsymbol{x}_1\neq\boldsymbol{x}_2}\frac{|\boldsymbol{\omega}(\boldsymbol{x}_1)-\boldsymbol{\omega}(\boldsymbol{x}_2)|}{|\boldsymbol{x}_1-\boldsymbol{x}_2|^\alpha}$$

Such solutions are said here to have "limited" regularity.

Blow up

A very smooth ride in a rough sea

- The limited-regularity solutions of Lichtenstein and his followers have a $C^{1,\alpha}$ regularity in space (their first derivatives are Hölder continuous). In Eulerian coordinates, their temporal regularity is also not better than $C^{1,\alpha}$. The Lagrangian structure, i.e. the trajectories of fluid particles, is however much smoother.
- Chemin (1982) showed that during the interval of regularity they remain *indefinitely differentiable* in time.
- Serfati (1995) and Shnirelman (2012), using the theory of analytic functions on complex Banach spaces, showed that the trajectories of fluid particles are actually analytic in time. Their non-constructive proofs are difficult and give no estimates of the radius of convergence of the time-Taylor series for the Lagrangan map.
- Frisch and Zheligovsky (2014), using Cauchy's formulation proved the following *Theorem*. Consider a space-periodic three-dimensional flow of incompressible fluid governed by the Euler equation. Suppose the initial velocity $\boldsymbol{v}_0(\boldsymbol{a})$ is in $C^{1,\alpha}(\mathbb{T}^3)$. Then, at sufficiently small times, the position of fluid particles, $\boldsymbol{x}(\boldsymbol{a},t)$, is given by a temporal Taylor series whose coefficients can be recursively calculated. The radius of convergence is bounded from below by a strictly positive quantity, which is inversely proportional to $|\nabla \boldsymbol{v}_0|_{0,\alpha}$.

$$\sum_{k=1}^{3} \nabla^{\mathrm{L}} \dot{x}_k \times \nabla^{\mathrm{L}} x_k = \boldsymbol{\omega}_0$$

$$\det(\nabla^{\mathrm{L}} \boldsymbol{x}) = 1$$

Proof Introduce the displacement: $\boldsymbol{\xi} := \mathbf{x} - \mathbf{a}$

$$\nabla^{\mathrm{L}} \times \dot{\boldsymbol{\xi}} + \nabla^{\mathrm{L}} \dot{\xi}_k \times \nabla^{\mathrm{L}} \xi_k = \boldsymbol{\omega}_0 \quad \det\left(\mathbf{I} + \nabla^{\mathrm{L}} \boldsymbol{\xi}\right) = 1 \quad \text{or} \quad \nabla^{\mathrm{L}} \cdot \boldsymbol{\xi} + \frac{1}{2}\left[\left(\nabla^{\mathrm{L}} \cdot \boldsymbol{\xi}\right)^2 - \mathrm{tr}\left(\nabla^{\mathrm{L}} \boldsymbol{\xi}\right)^2\right] + \det(\nabla^{\mathrm{L}} \boldsymbol{\xi}) = 0$$

Recursion relations: Expand in powers of t: $\boldsymbol{\xi} = \sum_{n=1}^{\infty} t^n \boldsymbol{\xi}^{(n)}$, and determine coefficients of various powers

$$n \nabla^{\mathrm{L}} \times \boldsymbol{\xi}^{(n)} + \sum_{r+s=n} r \nabla^{\mathrm{L}} \xi_k^{(r)} \times \nabla^{\mathrm{L}} \xi_k^{(s)} = \boldsymbol{\omega}_0 \delta_{n1}, \quad n = 1, 2, \ldots$$

$$\nabla^{\mathrm{L}} \cdot \boldsymbol{\xi}^{(n)} + \frac{1}{2} \sum_{r+s=n} \left[\nabla^{\mathrm{L}} \cdot \xi^{(r)} \, \nabla^{\mathrm{L}} \cdot \xi^{(s)} - \mathrm{tr}\left(\nabla^{\mathrm{L}} \xi^{(r)} \nabla \xi^{(s)}\right)\right] + \sum_{r+s+\sigma=n} \nabla^{\mathrm{L}} \xi_1^{(r)} \cdot \left(\nabla^{\mathrm{L}} \xi_2^{(s)} \times \nabla^{\mathrm{L}} \xi_3^{(\sigma)}\right) = 0$$

The beast and its taming

$$\begin{aligned} \boldsymbol{\nabla}^L_\mu \xi^{(s)}_\nu = {} & \boldsymbol{\nabla}^L_\mu v^{(init)}_\nu \delta^s_1 + \sum_{\substack{1\leqslant j\leqslant 3,\\ j\neq\nu}} \mathcal{C}_{\mu j} \left(\sum_{\substack{1\leqslant k\leqslant 3,\\ 0<m<s}} \frac{2m-s}{s} (\boldsymbol{\nabla}^L_\nu \xi^{(m)}_k)\, \boldsymbol{\nabla}^L_j \xi^{(s-m)}_k \right. \\ & + \mathcal{C}_{\mu\nu} \left(\sum_{\substack{1\leqslant i<j\leqslant 3\\ 0<m<s}} \left((\boldsymbol{\nabla}^L_j \xi^{(m)}_i) \boldsymbol{\nabla}^L_i \xi^{(s-m)}_j - (\boldsymbol{\nabla}^L_i \xi^{(m)}_i) \boldsymbol{\nabla}^L_j \xi^{(s-m)}_j \right) \right. \\ & \left. - \sum_{\substack{i,j,k\\ l+m+n=s}} \varepsilon_{ijk} (\boldsymbol{\nabla}^L_i \xi^{(l)}_1)(\boldsymbol{\nabla}^L_j \xi^{(m)}_2) \boldsymbol{\nabla}^L_k \xi^{(n)}_3 \right) . \end{aligned} \tag{2.29}$$

Here, $\mathcal{C}_{ij} \equiv \nabla^{-2} \boldsymbol{\nabla}^L_i \boldsymbol{\nabla}^L_j$ and $\boldsymbol{\nabla}^L_i \boldsymbol{\nabla}^L_j$ denotes the second-order partial derivative $\partial^2/\partial a_i \partial a_j$.

Proof of analyticity (continued)

$$
\begin{aligned}
\boldsymbol{\nabla}^L_\mu \xi^{(s)}_\nu = {} & \boldsymbol{\nabla}^L_\mu v^{(init)}_\nu \delta^s_1 + \sum_{\substack{1\leqslant j\leqslant 3,\\ j\neq\nu}} \mathcal{C}_{\mu j} \left(\sum_{\substack{1\leqslant k\leqslant 3,\\ 0<m<s}} \frac{2m-s}{s} (\boldsymbol{\nabla}^L_\nu \xi^{(m)}_k)\, \boldsymbol{\nabla}^L_j \xi^{(s-m)}_k \right) \\
& + \mathcal{C}_{\mu\nu} \left(\sum_{\substack{1\leqslant i<j\leqslant 3\\ 0<m<s}} \left((\boldsymbol{\nabla}^L_j \xi^{(m)}_i) \boldsymbol{\nabla}^L_i \xi^{(s-m)}_j - (\boldsymbol{\nabla}^L_i \xi^{(m)}_i) \boldsymbol{\nabla}^L_j \xi^{(s-m)}_j \right) \right. \\
& \left. - \sum_{\substack{i,j,k\\ l+m+n=s}} \varepsilon_{ijk} (\boldsymbol{\nabla}^L_i \xi^{(l)}_1)(\boldsymbol{\nabla}^L_j \xi^{(m)}_2) \boldsymbol{\nabla}^L_k \xi^{(n)}_3 \right) . \qquad (2.29)
\end{aligned}
$$

Here, $\mathcal{C}_{ij} \equiv \nabla^{-2} \boldsymbol{\nabla}^L_i \boldsymbol{\nabla}^L_j$ and $\boldsymbol{\nabla}^L_i \boldsymbol{\nabla}^L_j$ denotes the second-order partial derivative $\partial^2/\partial a_i \partial a_j$.

Define $\zeta_n := |\nabla \boldsymbol{\xi}^{(n)}|_{0,\alpha}$ and introduce the generating function $F(t) := \sum_{n=1}^{\infty} t^n \zeta_n$

Use the boundedness of $\Delta^{-1}\nabla\nabla$ in Hölder spaces (Korn 1907, Lichtenstein 1925, Stein 1970,...):

$$\zeta_n \leq C_1 |\mathbf{v}_0|_{1,\alpha} \delta_{n1} + C_2 \sum_{r+s=n} \zeta_r \zeta_s + C_3 \sum_{r+s+\sigma=n} \zeta_r \zeta_s \zeta_\sigma, \quad n = 1, 2, \ldots \quad C_1 > 0, \quad C_2 > 0, \quad C_3 > 0$$

$F(t) \leq C_1 t |\mathbf{v}_0|_{1,\alpha} + C_2 F^2(t) + C_3 F^3(t), \quad t > 0$ $F(t)$ bounded for $0 \leq t |\nabla \mathbf{v}_0|_{0,\alpha} < \tau_\star$

where $\tau_\star > 0$ is such that the discriminant $C_1 \tau_\star - F + C_2 F^2 + C_3 F^3 = 0$ vanishes.

This follows from the observation that the polynomial $P(F) := C_3 F^3 + C_2 F^2 - F$

has one finite local maximum and minimum and two positive roots, colliding on increasing t.

Analyticity follows from the boundedness of the generating function. QED

“Time”-analytic particle trajectories for flows with boundaries and for cosmological flow

- For incompressible flow, in the presence of a solid impermeable boundary, the recursion relations for the time-Taylor coefficients $\xi^{(n)}$ must be supplemented by conditions expressing the *invariance of the boundary under the Lagrangian flow*. As shown by Besse and Frisch (2017), this leads again to time-analyticity of Lagrangian trajectories, when the boundary is itself analytic (but there are interesting weaker results for Gevrey-class and ultra-differentiable boundaries). This is important for investigating *blow-up*, which may depend on the presence of boundaries.

- Cosmological flow, after matter-radiation decoupling and before the development of massively multi-streaming regimes, is governed by the gravitational Euler–Poisson equations for potential flow, coupled to the Friedmann equation for the expansion scale factor $a(t)$.
  For an Einstein–de Sitter universe, Zheligovsky and Frisch (2014) showed that analyticity in the scale factor a (used as a time variable) holds under certain slaving conditions. The result was extended to ΛCDM cosmology by Rampf, Villone and Frisch (2015).

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Lecture III. Beating CFL: the Cauchy-Lagrange numerical method

Eulerian vs. Lagrangian time-Taylor expansions

Eulerian (Euler 1755)

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p, \qquad \nabla\cdot\boldsymbol{v} = 0,$$

$$\partial_t v_i = \partial_j\left[v_i v_j - \nabla^{-2}\partial^2_{il}(v_l v_j)\right]$$

$(\nabla^{-2} =$ inverse Laplacian$)$

Lagrangian (Cauchy 1815)

$$\sum_{k=1}^{3} \nabla^{\mathrm{L}}\dot{x}_k \times \nabla^{\mathrm{L}} x_k = \boldsymbol{\omega}^{(\mathrm{init})}, \qquad \det(\nabla^{\mathrm{L}}\boldsymbol{x}) = 1,$$

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}, t), \quad \boldsymbol{x}(\boldsymbol{a}, 0) = \boldsymbol{a},$$

(Lagrangian map)

Eulerian Taylor expansion $\boldsymbol{v} = \sum_{s=0}^{\infty} \boldsymbol{v}^{(\mathrm{s})}\, t^s, \quad \boldsymbol{v}^{(0)} = \boldsymbol{v}^{(\mathrm{init})}$

$$s v_i^{(\mathrm{s})} = -\sum_{m+n=s-1} \partial_j\left[v_i^{(\mathrm{m})} v_j^{(\mathrm{n})} - \nabla^{-2}\partial^2_{il}\left(v_l^{(\mathrm{m})} v_j^{(\mathrm{n})}\right)\right]$$

Beauty?

Lagrangian Taylor expansion $\boldsymbol{x} = \boldsymbol{a} + \sum_{s=1}^{\infty} \boldsymbol{\xi}^{(\mathrm{s})} t^s$

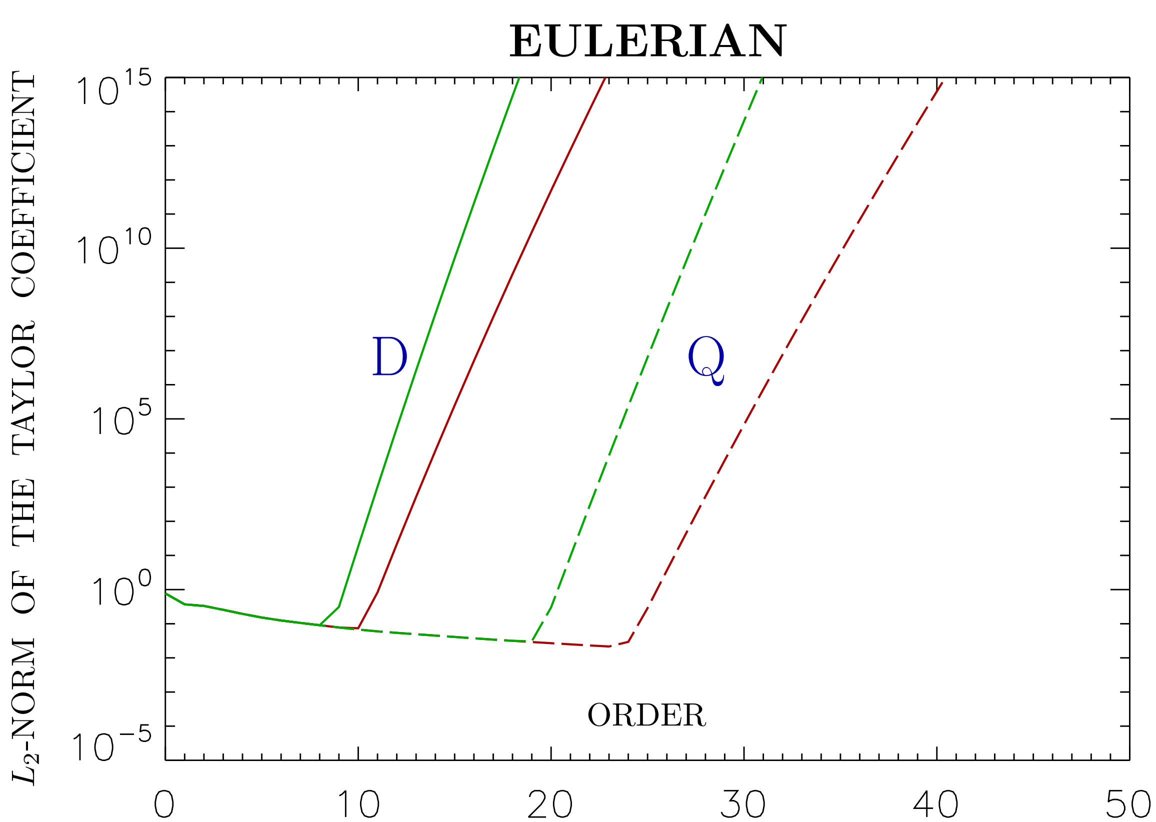
$$\begin{aligned}
\nabla^L_\mu \xi^{(s)}_\nu = {} & \nabla^L_\mu v^{(init)}_\nu \delta^s_1 + \sum_{\substack{1\leqslant j\leqslant 3,\\ j\neq\nu}} \mathcal{C}_{\mu j}\left(\sum_{\substack{1\leqslant k\leqslant 3,\\ 0<m<s}} \frac{2m-s}{s}\,(\nabla^L_\nu \xi^{(m)}_k)\,\nabla^L_j \xi^{(s-m)}_k\right) \\
& + \mathcal{C}_{\mu\nu}\left(\sum_{\substack{1\leqslant i<j\leqslant 3\\ 0<m<s}} \left((\nabla^L_j \xi^{(m)}_i)\nabla^L_i \xi^{(s-m)}_j - (\nabla^L_i \xi^{(m)}_i)\nabla^L_j \xi^{(s-m)}_j\right)\right. \\
& \left. - \sum_{\substack{i,j,k\\ l+m+n=s}} \varepsilon_{ijk}(\nabla^L_i \xi^{(l)}_1)(\nabla^L_j \xi^{(m)}_2)\nabla^L_k \xi^{(n)}_3\right).
\end{aligned} \tag{2.29}$$

Beast?

Loss of derivatives

Lagrangian time derivatives do not lose (spatial) derivatives (Ebin and Marsden 1970)

- In Eulerian coordinates, if $\mathbf{v} \in \mathrm{C}^{k,\alpha}$, then $\partial_t \mathbf{v} \in \mathrm{C}^{k-1,\alpha}$
- In Lagrangian coordinates, if $\mathbf{v} \in \mathrm{C}^{k,\alpha}$, then $D_t \mathbf{v} \in \mathrm{C}^{k,\alpha}$



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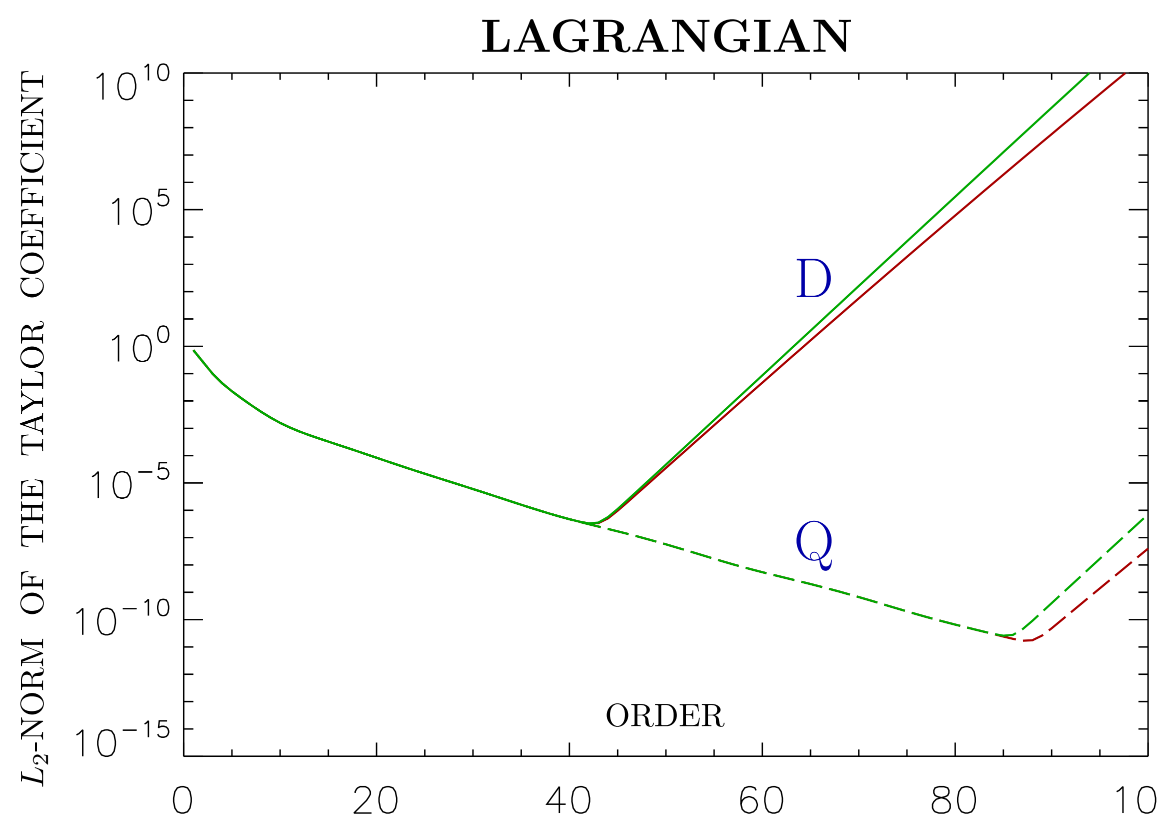


D: double, Q: quadruple precision. Resolution: 512^2, 1024^2

Eulerian computations: time step determined by CFL

- In 1928 Courant, Friedrichs and Lewy (CFL) showed that numerical solutions of hyperbolic PDE's by simple finite difference methods are subject to the constraint $\Delta t \, U_{\max}/\Delta x < 1$
- High-order Eulerian time-Taylor methods are subject to a similar CFL constraint, but for a very different reason related to rounding errors. For this, consider small-scale motion of amplitude ϑ advected by a quasi-uniform $\boldsymbol{U}$

  $\partial_t \vartheta + (\boldsymbol{U} \cdot \nabla)\vartheta = 0,$ Fourier decompose and solve

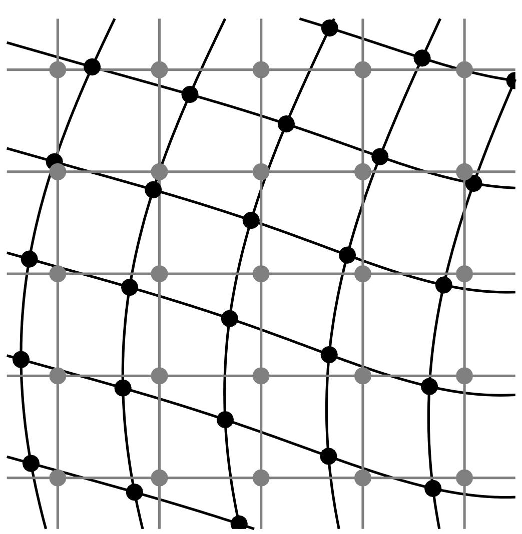
$$\vartheta_{\boldsymbol{k}}(t+\Delta t) = \vartheta_{\boldsymbol{k}}(t)\ \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{U}\Delta t) = \vartheta_{\boldsymbol{k}}(t) \sum_{s=0}^{\infty} \frac{(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{U}\Delta t)^s}{s!}$$

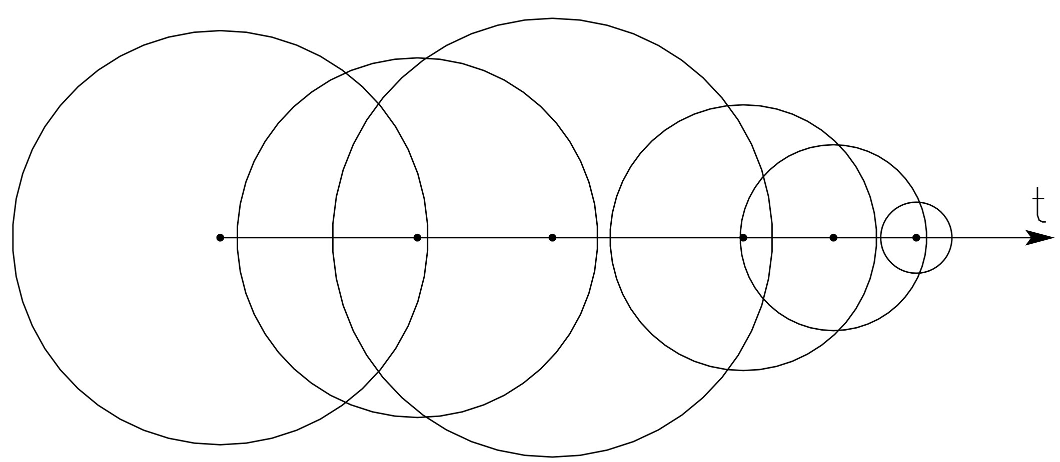
- If the Courant number $\mathrm{Co} \equiv k_{\max} U_{\max} \Delta t \gg 1$

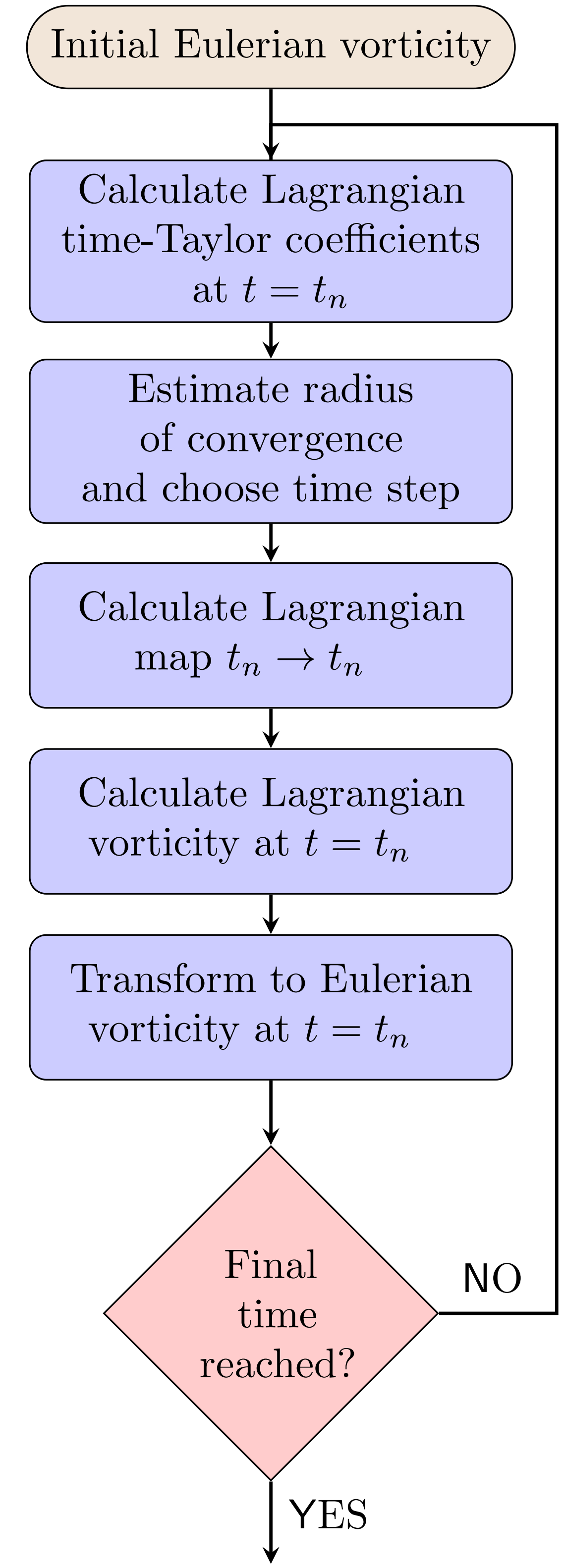
  there is a rounding noise problem because of cancellations

Lagrangian computations: time step determined by convergence radius

- Typically, the Lagrangian map is analytic but not entire in time: it has a finite radius of convergence, R (even in 2D).
- For any $0 < t < R$, one can construct a new time-Taylor series for a Lagrangian map, whose radius of convergence is $R(t)$.
- One can iterate this procedure and construct a sequence of increasing times $0 < t_1 < t < \ldots < t_m < \ldots$ This can be continued as long as $R(t_m)$ does not vanish.



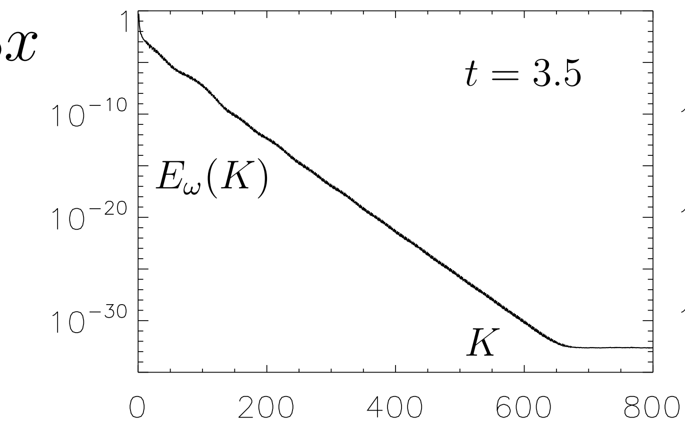


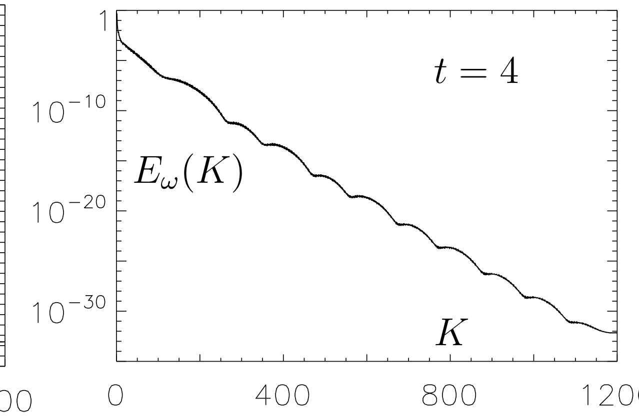


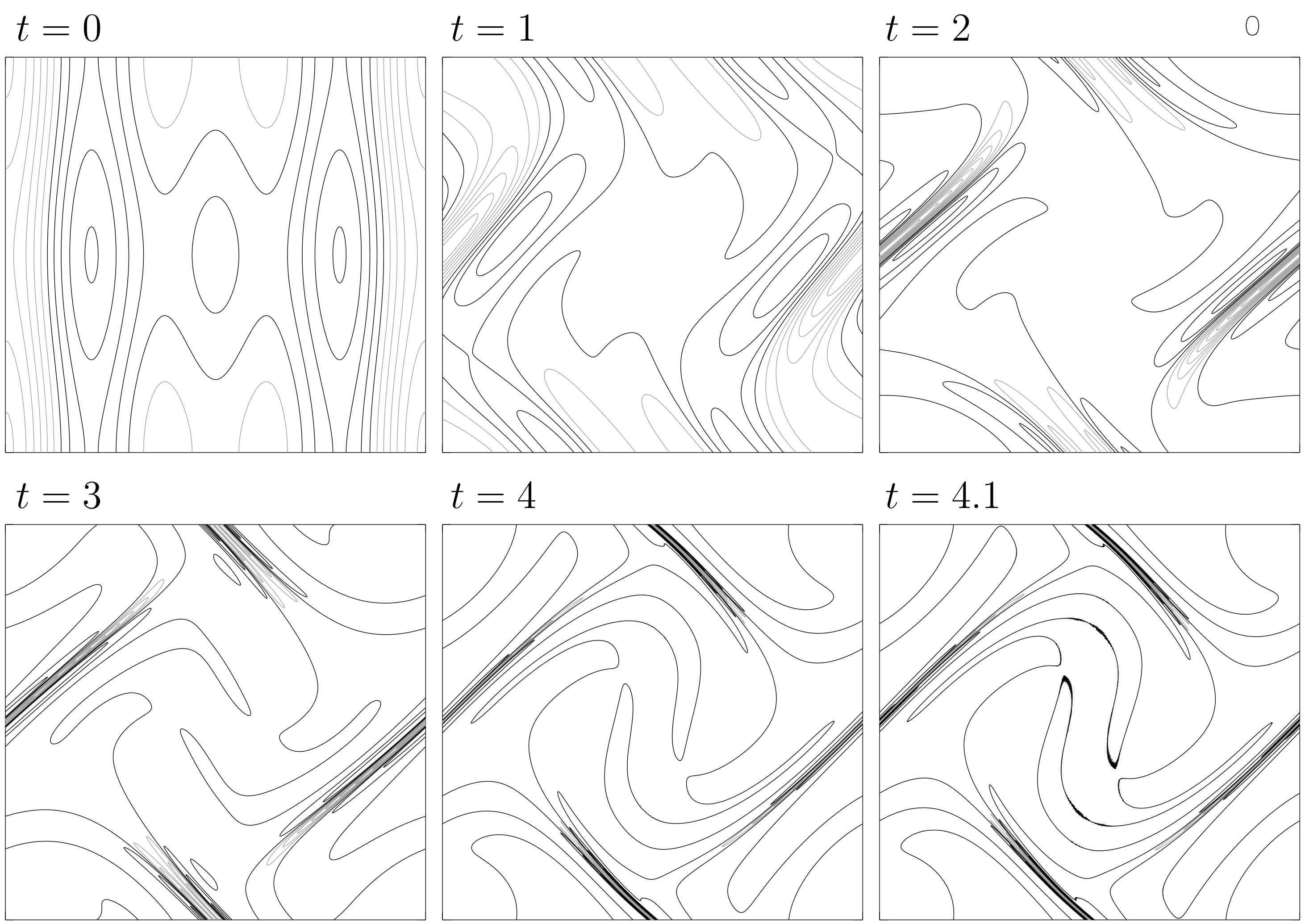
semi-Lagrangian

Switching from Eulerian to Lagrangian computations can result in speed up of several orders of magnitude

$$\boldsymbol{\omega}^{(\mathrm{init})} = \cos x + \cos y + 0.6\cos 2x + 0.2\cos 3x$$

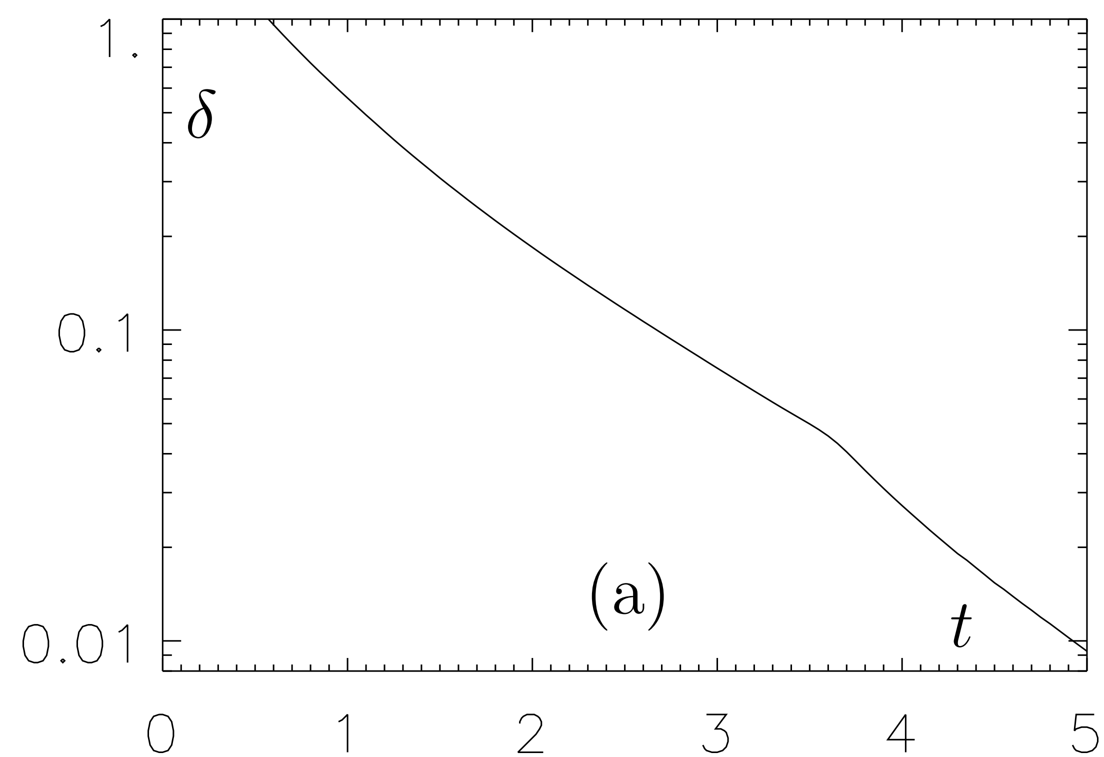






Isolines of the Laplacian of vorticity

Distance $\delta(t)$ to the real domain of the nearest complex-space singularity



Resolution: 8192^2 harmonics

Speed up CL20/RK4 : 120

3D Cauchy-Lagrange simulation by Akshay Bhatnagar

2015 PhD at IISc, Bangalore. Supervisor: Rahul Pandit

Resolution: 128^3; Time-Taylor expansion to order 14

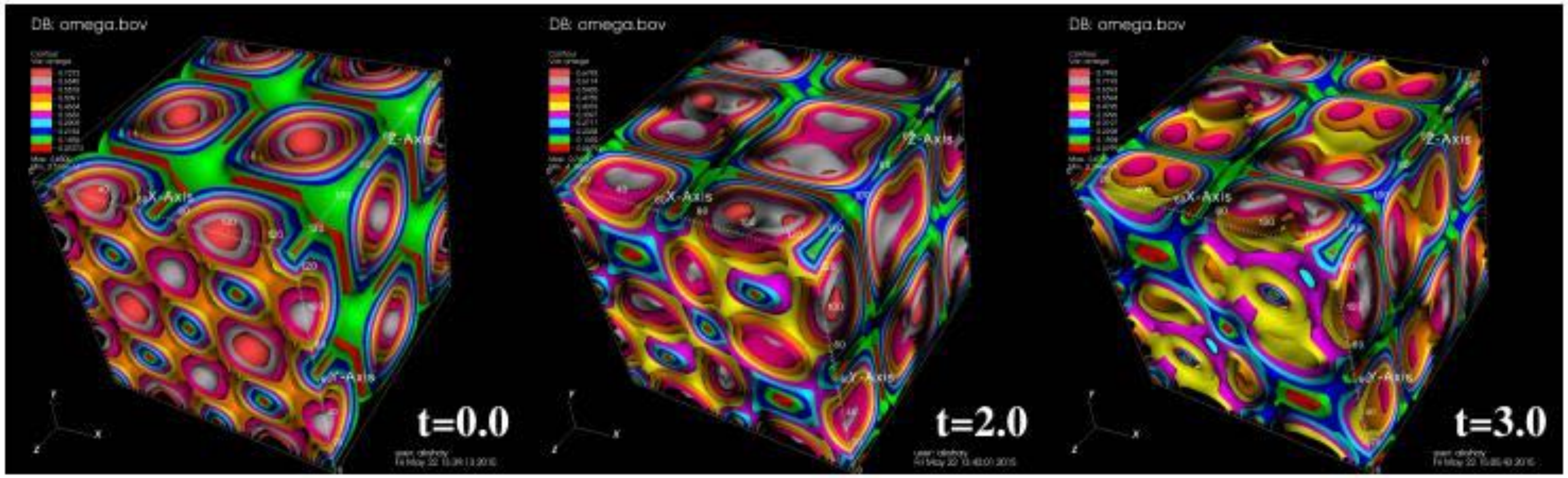


Fig. 6.5 Isosurfaces of the magnitude of the vorticity at 10 different levels, for three different

A 3D Taylor-Green flow is

$$u_x = \sin(x)\cos(y)\cos(z); \tag{6.20}$$

$$u_y = -\cos(x)\sin(y)\cos(z); \tag{6.21}$$

$$u_z = \cos(x)\cos(y)\sin(z). \tag{6.22}$$

This flow is a generalised version of the 2D Taylor-Green flow described in Section 6.3.3. In Fig. 6.5 we show the Isosurfaces of the magnitude of the vorticity at 10 different levels, for three different time instants, which we ob-

Conclusions and Perspectives

- Cauchy-Lagrange numerical methods, like other semi-Lagrangian methods, are meant to overcome the restriction of very small time steps, imposed by the CFL condition.
- Most semi-Lagrangian methods do not achieve high order in time and thus have limited accuracy, usually sufficient for, e.g., weather forecast.
- Thanks to the recursion relations that follow from Cauchy's 1815 Lagrangian formulation, the Cauchy-Lagrange method can easily achieve very high accuracy through the use of high-order time-Taylor expansions. This open new perspectives where accuracy is crucial, for example the study of *finite-time blow-up* in 3D.
- In 2D, the Cauchy-Lagrange method appears to have a substantial edge over Eulerian spectral methods at high resolution. In 3D the situation is not yet clear: so far the 3D Cauchy-Lagrange tests are about a factor four slower than an Eulerian spectral RK4 method. This may be due to the cubic nonlinearity of the "beast" and its double convolution structure.